27th EPS Conference on Contr. Fusion and Plasma Phys. Budapest, 12-16 June 2000 ECA Vol. **24B** *(2000) 868 -871*

Chaotic adiabatic dynamics in a 2D electrostatic field

Boris WEYSSOW[a] and Yves ELSKENS[b]
bweyssow@ulb.ac.be, elskens@newsup.univ-mrs.fr

[a]*Physique théorique et mathématique, Unité de physique des plasmas, Association Euratom-Etat Belge, Université Libre de Bruxelles, Campus plaine, CP 231, Bvd du Triomphe, B-1050 Bruxelles, Belgium*
[b]*Equipe turbulence plasma de l'UMR 6633 CNRS-Université de Provence, Physique des interactions ioniques et moléculaires, campus St-Jérôme, CP 321, F-13397 Marseille cedex 20, France*

The theory of 1.5 degrees-of-freedom hamiltonian systems permits to construct a "canonical action" which is increasingly better conserved as the time dependence of the hamiltonian is slow. This holds as far as the system undergoes no separatrix crossing. One is thus led to discard the case of a trajectory which escapes a basin of regularity (in KAM sense) determined by an elliptic point and bounded by separatrices associated with neighbor hyperbolic points.

In presence of separatrix crossings, the canonical action is discontinuous with respect to time. In the case of a system with slowly varying parameters, the successive discontinuities are strongly correlated. This observation suggests modifying the canonical action in such a way as to obtain a new function behaving continuously even at separatrix crossings. The existence of such "modified canonical action" is shown numerically. It leads to a criterion to determine the threshold for the loss of adiabaticity [W97, W98].

1. Introduction

Consider a 1.5D system, with a hamiltonian $H(x,y,\lambda)$ where x and y are canonically conjugate, $\lambda=\omega t$, and $H(x,y,\lambda)=H(x,y,\lambda+1)$. The canonical action is defined (up to an additive constant) for the integrable hamiltonian dynamics for fixed λ (frozen dynamics) [A76, LL69] by $2\pi S(x,y,\lambda)=\int_\Gamma x\,dy$ where Γ is the closed orbit of the frozen system passing through (x,y). In the adiabatic limit $d\langle S\rangle/dt=O(\omega)$. The condition $\omega\to 0$ actually requires that the period of the orbit with the same action in the frozen dynamics be small compared with the time scale of the variation of the hamiltonian. In the vicinity of separatrices of the frozen system, a specific separatrix-crossing neo-adiabatic theory is needed [CET86, H93, N86].

The canonical variable conjugate to the action is an angle θ, which is not defined at separatrix crossing. The crossing map, connecting the action-angle pair (S,θ) before and (S',θ') after crossing, shows that the angle is sensitive to the detailed crossing dynamics, which is the way chaos occurs in these hamiltonian systems [EE91, EE92, NST97], while the action is more robust. In this contribution we discuss how to better accommodate the definition of the action to separatrix crossings.

2. The three-waves hamiltonian system

Our work is motivated by the guiding center chaotic motion, induced by ExB drift in three electrostatic waves and a static and uniform magnetic field. We consider [W97, W98]

$$H(x,y,t) = \sin(2\pi\,(x+\omega\,t)) + \sin(2\pi\,(y+\omega\,t)) + \sin(2\pi\,(x+y+\omega\,t))\ .$$

Phase space is periodic in x (mod 1) and in y (mod 1). The time dependence in H leads to the periodic sweeping of square cells by two oblique separatrices.

Points (1/2 mod 1, 1/2 mod 1) are fixed for all frequencies ω. Lines x=1/2 mod 1 are invariant and connect heteroclinically neighboring fixed points ; the motion on them is one-directional, so that each fixed point is parabolic (neither elliptic nor hyperbolic). Similarly, lines y=1/2 mod 1 are also invariant, with one-directional motion on them. Therefore the fixed points are not directly responsible for the chaotic dynamics in the adiabatic limit, in contrast with the chaos related to slowly pulsating separatrices [EE91, EE92, NST97].

The fixed boundaries forbid large scale transport. The regularity of the square cells and of the sweeping separatrices motion is an advantage for this discussion as the chaotic transport is due only to separatrix crossings and not to exploration of various parts of phase space. Indeed the system is stochastic within each cell, and the motion of a guiding center in a square cell in the adiabatic limit is easily grasped. Oblique separatrices are crossed instantaneously, and S is constant (to order $\omega \ln\omega$) between crossings. At each crossing, $2\pi S$ takes the value of the area of the domain into which the trajectory gets captured. Only a domain with increasing area can capture the trajectory; if two domains have increasing area at the time of crossing, the probability for one of them to capture the trajectory is controlled by the rate at which its area is increasing.

These observations lead to a markovian probabilistic model for the sequence of separatrix crossings [W97, W98], which is satisfactory for $\omega<0.01$. However it is desirable to extend the model to faster frequencies.

3. Variation of the canonical action with phase and modified canonical action

Some insight on the evolution of the action during a separatrix crossing can be obtained from a simple argument. Consider the guiding centre motion in the vicinity of an elliptic point moving at constant velocity v along the x axis. Such a motion can be described by a quadratic hamiltonian with circular contour levels: $H(x,y;\lambda)=(x-\lambda)^2/2+y^2/2$ with λ=vt. Here, v is the velocity of displacement of the elliptic point localized at position (x,y)=(vt,0). A short time expansion of the analytic solution of the equations of motion gives : $x(\delta t) \approx x(0)+ y(0)\ \delta t - 1/2\ \delta t^2\ x(0)$, and $y(\delta t) \approx y(0) - x(0)\ \delta t - 1/2\ \delta t^2\ (y(0) - v)$. The squared distance at time δt of a point on the trajectory to the elliptic point is then $r^2(\delta t) = r^2(0) - 2\ x(0)\ v\ \delta t + O(\delta t^2)$. Assuming v > 0, an initial condition at the right of the elliptic point (x(0) > 0) yields at time δt a squared distance $r^2(\delta t) < r^2(0)$**.** Thus dS/dt<0 in this case. On the contrary, an initial condition at the left of the elliptic point ($x(0) < 0$) gives $r^2(\delta t) > r^2(0)$ and leads to dS/dt>0. Then after a time δt (fixed), $\delta S = (r^2(\delta t) - r^2(0))/2 \approx - x(0)\, v\, \delta t$. Thus the canonical action increases as the trajectory approaches the separatrix before crossing and decreases after crossing.

This mechanism can be retained for our three-wave system: for a majority of separatrix crossings, the relevant continuous function is obtained by changing the sign of the action obtained after the crossing and then by subtracting the jump. The time derivative of this continuous function is however mildly (logarithmically) divergent. In more general terms, the modified canonical action is obtained with or without changing the sign of the canonical action before removing the gap.

4. The modified canonical action in the three-waves system

We now show, using this rule, that the modified canonical action is a $C^{1-\varepsilon}$ function (with a $\omega \ln \omega$ defect in derivability). This property is verified numerically. Fig. 1a shows the time evolution of the canonical action over two separatrix crossings. Fig. 1b demonstrates the effect of changing the sign of the action in the intermediate region prior to the removal of the gaps. The oscillations of the two external branches indicate that the system is in the adiabatic regime.

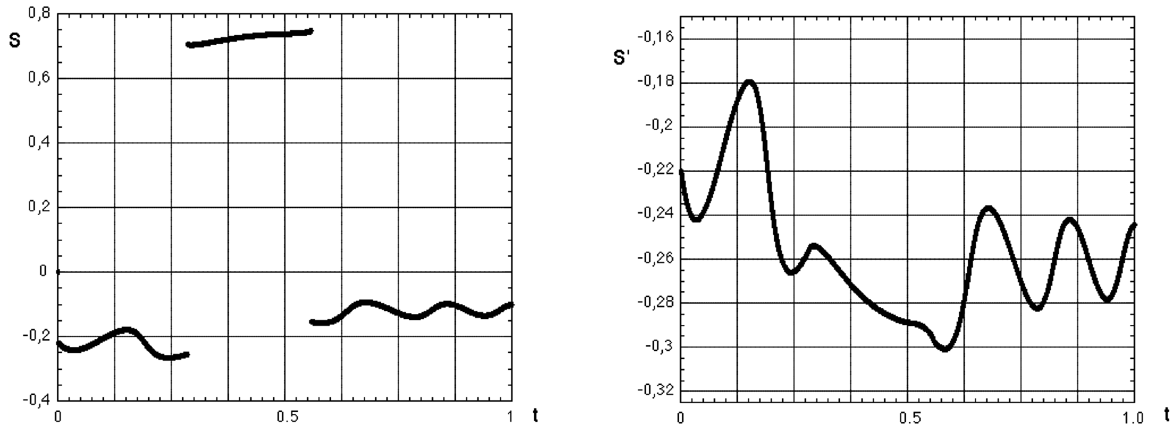

Fig. 1a (left): Evolution of canonical action S over two separatrix crossings, for ω=0.5.
Fig. 1b (right): Evolution of the modified canonical action.

We next consider a trajectory over a much longer time and at two different frequencies: ω=0.5 and ω=1. The behavior of the canonical action for ω=0.5 (Fig. 2a) shows that the dynamics is adiabatic : i) a time averaging can be defined, and ii) the mean value slowly varies in time.

At higher frequency ω=1 (Fig. 2b), the canonical action and the modified canonical action have a complicated behavior. It is then no more possible to define correctly a time averaging. This is typical of a non adiabatic behavior.

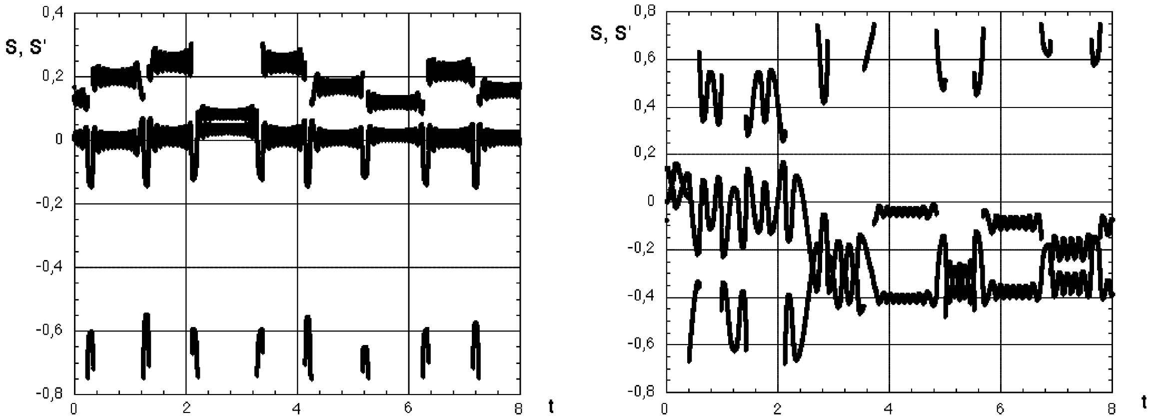

Fig. 2a : ω=0.5. Fig. 2b : ω=1.

Fig 2 : Canonical action S (in grey) and modified canonical action S' (in black) over eight periods.

5. Final remarks

The canonical action as well as the adiabatic invariant suffer from being defined differently on either sides of the separatrix crossing. In this work we introduced a technique to overcome the resulting singularity. It may be hoped that this technique will provide better insight to the stochastic transport in turbulent ExB drift.

Discussions with A.I. Neishtadt, D.F. Escande and F. Doveil are gratefully acknowledged. This work is partly supported by the belgian-french Tournesol cooperation program.

[A76] V. Arnold 1976, *Méthodes mathématiques de la mécanique classique* (Mir, Moscou).
[CET86] J.R. Cary, D.F. Escande & J.L. Tennyson 1986, **Phys. Rev. A 34**, 4256-4275.
[EE91] Y. Elskens & D.F. Escande 1991, **Nonlinearity 4**, 615-667.
[EE92] Y. Elskens & D.F. Escande 1992, **Physica D 62**, 66-74.
[H93] J. Henrard 1991, **Dynamics Reported 2**, 117-235 ; and references therein.
[LL69] L.D. Landau & E.M. Lifchitz 1969, *Mécanique* (Mir, Moscou).
[MN91] J.H. Misguich & R. Nakach 1991, **Phys. Rev. A 44**, 3869-3888.
[N86] A.I. Neishtadt 1986, **Fiz. Plasmy 12**, 992-1001; **Sov. J. Plasma Phys. 12**, 568-573.
[NST97] A.I. Neishtadt, V.V. Sidorenko & D.V. Treschev 1997, **Chaos 7**, 2-11.
[W97] B. Weyssow 1997, **Phys. Lett. A 232**, 234-242.
[W98] B. Weyssow 1998, **J. Plasma Phys. 59**, 761-772.
[WMR98] B. Weyssow, J.H. Misguich & J.D. Reuss 1998, **Phys. Rev. E 58**, 3768-3776.