

Spherical Electron Vortices

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Stationary propagating, spherical vortex solutions of the equations of electron magnetohydrodynamics are discussed. These electron vortices can be viewed as generalization of the Hill vortices in fluids. Electron spherical vortices in plasmas have been considered in Ref. [1] but no explicit solutions were given. Renewed interest in such vortices is due to their observation in plasma experiments [2].

In our study we use the model of electron magnetohydrodynamics (EMHD) [3, 4]. It describes phenomena at frequencies that are large compared to the ion gyro-frequency and smaller than the electron gyro- and plasma frequencies, $\omega_{Bi} \ll \omega \ll (\omega_{Be}, \omega_{pe})$. The characteristic perturbation scales l satisfy the relation $l \ll c/\omega_{pi}$, where ω_{pi} is the ion plasma frequency. Within this ordering, all ion motions are negligible and the dynamics is governed by the electrons only.

We consider a plasma in a strong equilibrium magnetic field that has its dominant component along the z -axis. We assume that the plasma motion is axisymmetric and neglect equilibrium gradients and perturbations of electron density and temperature. The magnetic field is represented as $\mathbf{B} = (B_0/r)(I\mathbf{e}_\theta + \nabla\chi \times \mathbf{e}_\theta)$. Here, \mathbf{e}_θ is the unit vector in the azimuthal direction, I is the axial current, χ is the axial magnetic flux, and (r, θ, z) are cylindrical coordinates. The unperturbed values of χ and I are taken to be $\chi \rightarrow r^2/2$ and $I \rightarrow j_0 r^2/2$, where $j_0 = (4\pi/cB_0)j_z$ is the normalized equilibrium current density. In this case, the equations of EMHD have the form [1]

$$\left(\frac{\partial}{\partial t} + \frac{\omega_{Be}d_e^2}{r}[\mathbf{e}_\theta \times \nabla I] \cdot \nabla\right) \left(\frac{I - d_e^2\Delta I}{r^2}\right) = -\frac{\omega_{Be}d_e^2}{r}[\mathbf{e}_\theta \times \nabla(\chi - d_e^2\Delta\chi)] \cdot \nabla\left(\frac{\Delta\chi}{r^2}\right),$$

$$\left(\frac{\partial}{\partial t} + \frac{\omega_{Be}d_e^2}{r}[\mathbf{e}_\theta \times \nabla I] \cdot \nabla\right)(\chi - d_e^2\Delta\chi) = 0, \quad (1)$$

where $d_e = (mc^2/4\pi ne^2)^{1/2}$ is the electron inertia skin depth and Δ denotes the operator $\Delta = r(\partial/\partial r)r^{-1}(\partial/\partial r) + \partial^2/\partial z^2$.

We study perturbations that propagate in the z -direction with uniform velocity u_z and that depend on space-time through the variables r and $\hat{z} = z - u_z t$. We introduce the normalized variables $t \rightarrow \omega_{Be}t$, $(r, \hat{z}) \rightarrow (r, \hat{z})/d_e$, $u_z \rightarrow u_z/(d_e\omega_{Be})$, $I \rightarrow d_e I$, and $\chi \rightarrow d_e^2\chi$. Equations (1) have general solutions of the form [1]

$$\chi - \Delta\chi = F_\chi(\hat{I}), \quad I - \Delta I = -F'_\chi(\hat{I})\Delta\chi + r^2 F_I(\hat{I}), \quad (2)$$

where F_χ and F_I are arbitrary functions of $\hat{I} \equiv I + u_z r^2/2$. Following Ref. [1] we take F_χ to be a linear function of its argument and $F_I \equiv C_I$ a piece-wise constant. The values of $C_\chi \equiv F'_\chi$ and C_I at infinity are determined from the boundary conditions, $C_\chi^{(\infty)} = 1/(j_0 + u_z)$ and $C_I^{(\infty)} = j_0/2$. Nonsingular solutions of Eqs (2) do not exist if both C_χ and C_I have the same values over whole space. In order to avoid singularities a surface must exist, the separatrix, that separates regions where the constants C_χ and C_I have different values. We suppose that the separatrix is a sphere with radius R_s where C_χ changes while $C_I \equiv C_I^{(\infty)}$. To provide continuity of ΔI and $\Delta\chi$, the functions I and χ must vanish at the separatrix. In spherical coordinates, $R = (r^2 + \hat{z}^2)^{1/2}$, $\phi = \arctan(r/\hat{z})$, solutions to Eqs (2) that satisfy this requirement are

$$\begin{aligned} \hat{I} &= -\frac{j_0 + u_z}{2} \left\{ \frac{1}{\hat{k}_1^{(a)} + \sigma^{(a)} \hat{k}_2^{(a)}} \left[\sigma^{(a)} \hat{k}_2^{(a)} \frac{b_1(\hat{k}_1^{(a)} \rho)}{b_1(\hat{k}_1^{(a)})} + \hat{k}_1^{(a)} \frac{b_1(\hat{k}_2^{(a)} \rho)}{b_1(\hat{k}_2^{(a)})} \right] - \rho \right\} \rho R_s^2 \sin^2 \phi, \\ \chi &= i \frac{j_0 + u_z}{2(\hat{k}_1^{(a)} + \sigma^{(a)} \hat{k}_2^{(a)})} \left\{ \hat{k}_2^{(a)2} \frac{b_1(\hat{k}_1^{(a)} \rho)}{b_1(\hat{k}_1^{(a)})} - \hat{k}_1^{(a)2} \frac{b_1(\hat{k}_2^{(a)} \rho)}{b_1(\hat{k}_2^{(a)})} + (\hat{k}_1^{(a)2} - \hat{k}_2^{(a)2}) \rho \right\} \rho R_s \sin^2 \phi. \end{aligned} \quad (3)$$

Here, the superscript $a = (i, e)$ denotes internal (inside the separatrix) and external (outside the separatrix) regions, respectively, $\rho = R/R_s$ is the normalized radius, $\sigma^{(a)} = \pm 1$ and $\hat{k}_j^{(a)}$ are the vortex eigen values determined by $\hat{k}_{1,2}^{(a)} = (R_s/2)/\{2 - C_\chi^2 \pm [(2 - C_\chi^2)^2 - 4]^{1/2}\}$. The values of $\hat{k}_{1,2}^{(a)}$ satisfy the condition $\hat{k}_1^{(a)} \hat{k}_2^{(a)} = \sigma^{(a)} R_s^2$ and they can be either pure imaginary or complex conjugated. Localized solutions exist if $\text{Re}(k_j^{(e)}) > 0$ which implies $\sigma^{(e)} = 1$. Matching the functions \hat{I} , χ and their derivatives at the separatrix $R = R_s$, one finds that $\hat{k}_j^{(a)}$ should satisfy the dispersion relation

$$(\hat{\kappa}_1 + \sigma^{(i)} \hat{\kappa}_2) \frac{K_{5/2}(\hat{\kappa}_1)}{K_{3/2}(\hat{\kappa}_1)} + (\hat{\kappa}_1 - i\sigma^{(i)} \hat{\kappa}_2) \frac{J_{5/2}(\hat{\kappa}_1)}{J_{3/2}(\hat{\kappa}_1)} + \sigma^{(i)} (\hat{\kappa}_1 + i\hat{\kappa}_1) \frac{J_{5/2}(\hat{\kappa}_2)}{J_{3/2}(\hat{\kappa}_2)} = 0, \quad (4)$$

where $\hat{k}_1 \equiv \hat{k}_1^{(e)}$, $\hat{k}_j \equiv i\hat{k}_j^{(i)}$, and K and J are spherical Bessel functions.

Two different cases exist, $\hat{\kappa}_1 \hat{\kappa}_2 = -\sigma^{(i)} R_s^2 = \mp R_s^2$. The case $\sigma^{(i)} = -1$ reduces to the case $\sigma^{(i)} = 1$ by the substitution $(\hat{\kappa}_1, \hat{\kappa}_2) \rightarrow (-\hat{\kappa}_2, \hat{\kappa}_1)$. Thus, without loss of generality we take $\sigma^{(i)} = 1$. The external \hat{k}_1 is written in the form $\hat{k}_1 = R_s \exp(i\alpha)$. For every value of α this leads to a number of roots consisting of combinations of $(\hat{\kappa}_1, \hat{\kappa}_2)$. The dispersion relation (4) has singular points at the roots of the Bessel functions. For that reason we investigate the roots of the equation that is obtained after multiplication with $K_{3/2}(\hat{\kappa}_1) J_{3/2}(\hat{\kappa}_1) J_{3/2}(\hat{\kappa}_2)$. This new equation has the same roots as the original one but also has extra roots where $J_{3/2}(\hat{\kappa}_{1,2}) = 0$. The real (solid) and imaginary (dashed) parts of this relation for $\alpha = 0.5$ are shown in Fig. 1 (a).

The figure shows all four quadrants. The roots with $\hat{\kappa}_1 \hat{\kappa}_2 = -R_s^2$ lie in the second and fourth quadrant. Moreover there is symmetry with respect to the line $\hat{\kappa}_2 = -\hat{\kappa}_1$. This follows directly from $J_{5/2}(-x)/J_{3/2}(-x) = -J_{5/2}(x)/J_{3/2}(x)$. From Eqs (3) it is readily seen that the solutions for \hat{I} and χ are the same at such corresponding points. Consequently we direct our attention to the second and fourth quadrants below the line $\hat{\kappa}_2 = -\hat{\kappa}_1$.

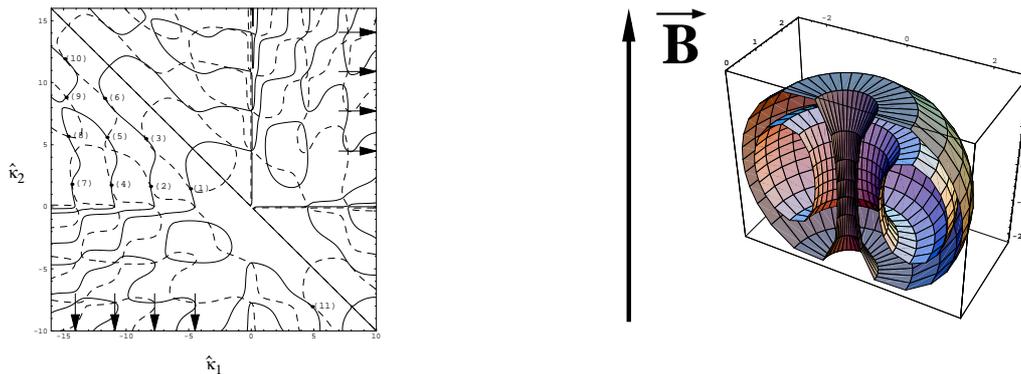


Fig. 1. (a) Roots of the dispersion relation (4) for $\alpha = 0.5$ and (b) spatial structure of the spherical vortex that corresponds to the root marked as (1).

The points of intersection of the two sets of curves are solutions of Eq. (4) except for those that correspond to $J_{3/2}(\hat{\kappa}_{1,2}) = 0$, indicated in Fig. 1 (a) by arrows. Actual solutions of Eq. (4) are marked by numbers (1), (2), . . . (11). The three dimensional structure of the vortex corresponding to solution (1) is presented in Fig. 1 (b). Surfaces where \hat{I} and χ are constant form a family of nested tori. Figure 1 (b) shows surfaces where $2\hat{I}/(j_0 + u_z) = 0.5$ (inner surface) and 5.0 (outer surface). This vortex has only a single null surface, the separatrix, where the potentials \hat{I} and χ vanish.

Roots that correspond to vortices with as few as possible null surfaces are of special interest. Such vortices are expected to be the most stable ones. In Fig. 2 the functions $-2\hat{I}/R_s^2(j_0 + u_z)$ and $-2i\chi/(j_0 + u_z)$ with $\phi = \pi/2$ (i.e. the expressions between curly brackets in Eqs (3)) have been plotted for the roots that are marked in Fig. 1 (a) as (1), (2) and (4). These solutions show that \hat{I} as well as χ develop additional zeros for larger values of $|\hat{\kappa}_1|$. The only root where \hat{I} and χ have zeros at $R = 0$ and R_s is (1). Figure 3 shows how the radial behaviour of \hat{I} and χ depends on the vortex eigen values $\hat{\kappa}_j^{(e)}$ in the external region. It presents solutions for \hat{I} and χ at $\phi = \pi/2$ at different values of α . This figure demonstrates that the maximum value of χ grows more rapidly with α than that of \hat{I} . It also shows that neither function develops an extra zero inside the separatrix $0 \leq R \leq R_s$. However for large values of α they do form extra zeros outside $R = R_s$.

To conclude, we have derived a general dispersion relation for spherical electron vortices in magnetized plasmas and expressions that describe their spatial structure. Vortices that we have found analytically look very similar to those observed in experiments by Stenzel et al. [2].

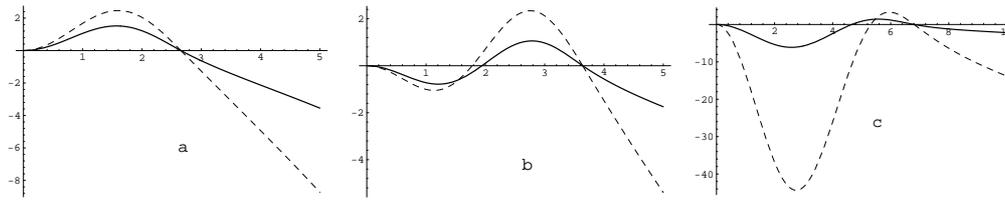


Fig. 2. Normalized potentials \hat{I} (solid lines) and χ (dashed lines) as functions of R for $\alpha = 0.5$. Figures a, b, c correspond to roots marked as (1), (2), (4) in Fig. 1 (a).

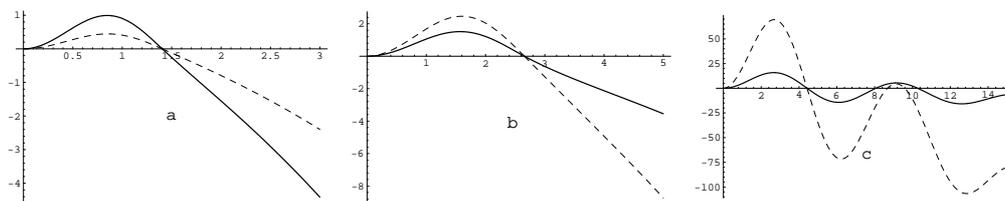


Fig. 3. Dependence of \hat{I} (solid lines) and χ (dashed lines) on R ; (a) $\alpha = 0.2$, (b) $\alpha = 0.5$, (c) $\alpha = 1.5$.

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