Isothermal internal kink instability in a toroidally rotating tokamak plasma

C. Wahlberg[1] and A. Bondeson[2]

[1]*Department of Space and Plasma Physics, EURATOM/NFR Fusion Association, Uppsala University, S-755 91 Uppsala, Sweden.*

[2]*Department of Electromagnetics, EURATOM/NFR Fusion Association, Chalmers University of Technology, S-412 96 Göteborg, Sweden.*

The stability properties of tokamak plasmas with equilibrium mass flows have attracted considerable attention in recent years. However, one difficulty with equilibria with mass flow is that the stability properties often are found to depend critically on the equation of state being used. One example is the internal kink instability in a toroidally rotating tokamak plasma. In a recent study [1] of this instability it was found that sonic toroidal rotation of the plasma transforms the internal kink instability into a stable oscillation (if $\Gamma > 1$), with the Doppler-shifted frequency

$$\omega_D^2 = \frac{\Omega^2 M^2}{3}\left(1 - \frac{1}{\Gamma}\right). \tag{1}$$

Here, Ω is the rotational frequency, M is the sonic Mach number and Γ is the adiabatic index. Furthermore, Eq. (1) represents an approximate form, valid for $M \ll 1$, of a more general expression for ω_D, valid for arbitrary M [1]. The character of the oscillations described by Eq. (1) is similar to the so-called Brunt-Väisälä oscillations of a stably stratified fluid in a gravitational field. In terms of the inverse aspect ratio $\varepsilon = r_0/R_0$, where r_0 is the minor radius and R_0 the major radius, the oscillation frequency ω_D above is a factor $1/\varepsilon$ larger than the Bussac growth rate γ_B [2], provided that $\Omega \sim \omega_s$ and $M \sim 1$, where ω_s is the sound frequency. With the ordering $\Omega/\omega_s \sim M \sim \varepsilon^{1/2}$, however, the internal kink drive is able to compete with the Brunt-Väisälä stabilisation, and the condition for stability becomes $\omega_D > \gamma_B$ in this regime [1, 3].

It is seen in Eq. (1) that the Brunt-Väisälä stabilisation of the internal kink disappears if an isothermal equation of state ($\Gamma = 1$) is assumed. There are, however, arguments for choosing $\Gamma = 1$ in the case of rotating plasmas [4, 5]. This is particularly relevant for frequencies much smaller than the sound frequency, for instance near marginal stability. The present paper therefore examines the consequences of using an isothermal MHD model for this instability.

The ideal, internal kink mode with toroidal mode number $n = 1$ in a toroidally rotating plasma is described by the coupling of the $m = 1$ amplitude ξ_1 (m is the poloidal mode number) to the $m = 2$ amplitude ξ_2 according to the equations [1]

$$L_1(\xi_1)+\varepsilon^2\left[T_1(\xi_1)+W_1\xi_1+\frac{d}{dr}\left(r^3W_2\frac{d\xi_1}{dr}\right)+r^2W_3\frac{d\xi_1}{dr}+\frac{d}{dr}\left(r^3W_4\frac{d\xi_2}{dr}\right)+\frac{d}{dr}\left(r^2W_5\xi_2\right)\right]=0, \tag{2a}$$

$$L_2(\xi_2)+\frac{d}{dr}\left(r^3U_1\frac{d\xi_1}{dr}\right)+r^2U_2\frac{d\xi_1}{dr}=0. \tag{2b}$$

Here it has been assumed that the equilibrium has i) large aspect ratio, ii) circular cross section, iii) $\beta_p \sim 1$, and iv) $\Omega \sim \omega_s \sim \varepsilon\omega_A$. The operators L_m and T_1 in Eqs. (2) are defined as

$$L_m \equiv \frac{d}{dr}\left[r^3(m\mu-1)^2\frac{d}{dr}\right]-r\left(m^2-1\right)(m\mu-1)^2,\quad T_1\equiv\frac{d}{dr}\left(r^3A_1\frac{d}{dr}\right)+r^2\frac{dA_2}{dr} \tag{3, 4}$$

$$A_1=-\rho\omega_D^2R_0^2+\rho\Omega^2M^2R_0^2-\frac{\left(2\omega_D^2-\omega_D\Omega+\Omega^2/4\right)(\rho\Omega)^2R_0^4+(\omega_D-2\Omega)\rho\omega_D\Gamma pR_0^2}{\Gamma p-\rho\omega_D^2R_0^2}$$

$$-\frac{\left(2\omega_D^2+(2\mu-1)\omega_D\Omega+(2\mu-1)^2\Omega^2/4\right)(\rho\Omega)^2R_0^4+\left(\omega_D+2(2\mu-1)\Omega\right)\rho\omega_D\Gamma pR_0^2}{(2\mu-1)^2\Gamma p-\rho\omega_D^2R_0^2}, \tag{5a}$$

$$A_2=-\rho\omega_D^2R_0^2-\frac{\left(3\omega_D^2-2\omega_D\Omega+\Omega^2/2\right)(\rho\Omega)^2R_0^4+\left(2\omega_D^2-4\omega_D\Omega+\Omega^2\right)\rho\Gamma pR_0^2}{\Gamma p-\rho\omega_D^2R_0^2}. \tag{5b}$$

Here, $\mu \equiv q^{-1}$ and ω_D is the Doppler-shifted mode frequency, assumed to be of the same order of magnitude as Ω. The coefficients W_n and U_n in Eqs. (2a,b) are given in Ref. [1]. The Shafranov-shift is given by $\Delta' \equiv d\Delta/dr = -(r/R_0)(\beta_p + \ell_i/2)$, where the inductance ℓ_i is defined in the same way as for $\Omega \equiv 0$ [2], whereas β_p is modified by the rotation [3]:

$$\beta_p=-\frac{2R_0^2}{\mu^2r^4}\int_0^r r'^2\frac{d}{dr'}\left(p+M^2p\right)dr'. \tag{6}$$

In the case of a monotonically increasing safety factor, with $q_0 < 1$, the lowest order solution of Eq. (2a) is given by $\xi_1 = \text{const.} = \ddot{\xi}$ in the region $0 \le r < r_1$, and $\xi_1 \equiv 0$ for $r > r_1$, where $\mu(r_1) = 1$. Integration of Eq. (2a) from $r = 0$ to $r = r_1 - 0$ gives

$$(\mu-1)^2r^3\left.\frac{d\xi_1}{dr}\right|_{r=r_1-0}=-\varepsilon^2\left(r^3A_1\left.\frac{d\xi_1}{dr}\right|_{r=r_1-0}+\ddot{\xi}\int_0^{r_1-0}r^2\frac{dA_2}{dr}dr+\ddot{\xi}\int_0^{r_1-0}W_1dr+r^3W_2\left.\frac{d\xi_1}{dr}\right|_{r=r_1-0}\right.$$

$$\left.+r^3W_4\left.\frac{d\xi_2}{dr}\right|_{r=r_1-0}+r^2W_5\xi_2\big|_{r=r_1-0}\right). \tag{7}$$

The condition that the right hand side of this equation is free from discontinuities at the resonant surface $r = r_1$ leads to the requirement $A_1(r_1) = 0$. This condition determines, to leading order, the eigenfrequency of the $m = n = 1$ mode, given by Eq. (1) in the case of weak rotation.

For an isothermal plasma we substitute $\Gamma = 1$ and $\mu = 1$ into the coefficient A_1:

$$A_1(r_1) = -\frac{\rho\omega_D^2 R_0^2\left(pF(M) - \rho\omega_D^2 R_0^2\right)}{p - \rho\omega_D^2 R_0^2}, \tag{8}$$

where $F(M) = 3 + 8M^2 + 2M^4$. Apart from the high-frequency solution $\omega_D = (pF/\rho)^{1/2}/R_0$, the equation $A_1(r_1) = 0$ now appears to lead to the usual Bussac root [2] $\omega_D \sim \varepsilon\omega_s \sim \varepsilon^2\omega_A$, with some enhancement of the inertia from the rotation, through the factor $F(M)$. However, by inspection of the coefficients in Eq. (7), it is seen that in this case there is also a contribution from the coefficient A_2, provided that $M \sim 1$. With $\omega_D \sim \varepsilon$ and $\Gamma = 1$, we obtain, to leading order, $A_2 = -\rho\Omega^2 R_0^2(1 + M^2) + O(\varepsilon)$. A convenient quantity representing this rotational contribution to the internal kink stability is the "rotational beta value"

$$\beta_\Omega = -\frac{2R_0^2}{\mu^2 r^4}\int_0^r r'^2 \frac{d}{dr'}\left(u_\Omega + M^2 u_\Omega\right)dr', \tag{9}$$

where $u_\Omega = \rho\Omega^2 R_0^2/2$ denotes the kinetic energy density of the rotation. Notice the similarity with the definition of β_p in Eq. (6). Introducing the inertial layer scaling $x = (r - r_1)/\varepsilon^2$ into the coefficients in Eq. (7), we get the layer equation ($\omega_D^2 \to -\gamma_D^2$)

$$\left[(q'(r_1)x)^2 + \rho\gamma_D^2 R_0^2 F(M)\right]\frac{d\xi_1}{dx} = \frac{r_1\hat{\xi}}{R_0^2}\left(\delta W_B - \beta_\Omega\right), \tag{10}$$

where δW_B is Bussac's *et al.* expression for the potential energy of the $m = n = 1$ mode [2]. Matching the solution to Eq. (10) to the lowest-order expression for ξ_1 as $x \to \pm\infty$, we obtain the growth rate of the isothermal, internal kink instability of a rotating tokamak plasma as

$$\gamma_D = -\frac{\pi r_1(\delta W_B - \beta_\Omega)}{q'(r_1)\left(\rho F(M)\right)^{1/2} R_0^3}. \tag{11}$$

In the case $\Delta q = 1 - q_0 \ll 1$, the quantity δW_B has the property $\delta W_B \propto \Delta q$, independently of the current profile [2]. The value of β_Ω, on the other hand, is finite as $\Delta q \to 0$. This is similar to the contributions from, e.g., ellipticity [6] and trapped particles [7], which do not contain Δq. The presence of such Δq-independent terms for realistic equilibria means that δW_B is relatively unimportant for small values of Δq. When the kinetic energy density u_Ω increases with r, we see that, for a decreasing pressure with the minor radius, the function $u_\Omega(1 + M^2)$

also increases with r, implying that $\beta_\Omega < 0$ and the rotation is strongly stabilising. In the opposite, and more realistic, situation with u_Ω decreasing with r, the sign of β_Ω depends on how much the pressure profile is peaked compared to the peaking of the u_Ω -profile. Assuming that the $q = 1$ radius is not too large, and introducing the length scales L_p and L_u for the radial variation of the pressure and kinetic energy density, respectively, $u_\Omega(r) = u_0[1 - (r/L_u)^2)]$, $p(r) = p_0[1 - (r/L_p)^2)]$, we find that the condition for a negative value of β_Ω becomes

$$\left(\frac{L_u}{L_p}\right)^2 > 2 + \frac{1}{M^2(0)} . \tag{12}$$

At sufficiently small L_p compared to L_u, rotation accordingly stabilises the isothermal, internal kink also in the case of a peaked kinetic energy density profile. Notice, however, that the condition (12) is rather restrictive, and is not satisfied for normal profiles.

In order to quantify the stability condition $\delta W_B - \beta_\Omega > 0$ in the case of low shear, we assume for simplicity that $L_p \approx L_u$, implying that $M(r) \approx$ constant. In this case the rotation is strongly destabilising and we have that $\beta_\Omega \approx M^2 \beta_p$. Assuming that $\Delta q = 1 - q_0 << 1$, the stability condition for a parabolic current profile becomes

$$\delta W_B - \beta_\Omega \approx \Delta q\left(\frac{13}{48} - 3\beta_p^2\right) - M^2 \beta_p > 0 . \tag{13}$$

If we also assume that $M^2 >> \Delta q$, the critical β_p from this condition becomes very small, and we get in a first approximation $\beta_p < 13\Delta q/(48 M^2) << 1$ for stability.

There are some similarities between the results found here and the results obtained by Waelbroeck [3]. For instance, the inertia enhancement factor $F(M)$ in Eq. (8) is similar to the expression $\nu(\omega_D)$ in Ref. 3. Furthermore, Waelbroeck defines a quantity (W_{Erot}) similar to the rotational beta value in Eq. (9). We emphasise, however, that these terms are of importance only in the case of an isothermal equation of state, $\Gamma = 1$. If $\Gamma > 1$, the contributions from the rotational terms investigated here are smaller by one order of magnitude than the Brunt-Väisälä stabilisation effect in Eq. (1). The proper value of Γ cannot, of course, be determined from the present, fluid theory. A kinetic description of the parallel particle motion would be necessary in order to do that.

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