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Non-linear wave interaction in a Hamiltonian family of Hasegawa-Mima related equation

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Abstract

A family of equations related to the Hasegawa-Mima equation are shown to be Hamiltonian. As an application of the Hamiltonian theory we present results concerning non-linear wave interaction in presence of a vortex street.

1. Introduction

One of the simplest equations that admit localized vortex structures is the Hasegawa-Mima (HM) equation describing a non-uniform low frequency plasma. Due to its simple form the HM-equation has been studied extensively both numerically (e.g. [1]) and analytically (e.g. [2]). In addition to vortex solutions the HM-equation contains drift waves which are a result of the non-uniform density.

In the weak turbulence approach (e.g. [3]) three of these linear modes interacts resonantly. In strong turbulence, however, a more accurate description would be a superposition of linear modes and vortex structures (e.g. [4]). Since both linear waves and nonlinear vortex structures exist in the HM-equation, a study of plasma turbulence (e.g. vortex-wave interaction) in the frame of this equation is natural.

Turbulence in fusion devices produce anomalous transport losses which diminish the confinement efficiency. It has, however, been shown that the anomalous transport can greatly be reduced by creation of strong sheared flows in the plasma (see e.g. the review paper [5]).

In the present paper a generalized form of the HM-equation is used to study the interaction between drift waves in presence of a sheared flow in the form of a vortex street. In a reference frame where the vortex street is stationary we use Hamiltonian theory [6] to study nonlinear wave interactions.

2. Basic equations

Let us consider a plasma with an inhomogeneous background. The unperturbed variables are the plasma density $n_0(y)$, the electron temperature $T(y)$ and the external magnetic field $B_0\hat{\mathbf{z}}$. We also assume quasi-neutrality and that the electrons are in Boltzman equilibrium. The analogy between this plasma system and a quasi-two-dimensional rotating fluid has been discussed in several papers (e.g. [7,8]) and the result is that we can use one model equation to represent both system.

The basic model equation which we will use is a generalization of the HM-equation

$$(\triangle_\mu \psi)_t + \{\psi, \triangle_\mu \psi + f\} = 0 \tag{1}$$

where $\mu(x, y)$ and $f(x, y)$ are arbitrary functions, $\triangle_\mu \equiv \triangle - \mu^2(x, y)$ and $\{a, b\} \equiv \partial_x a \partial_y b - \partial_x b \partial_y a$. Here ψ can be either the electric potential used in the plasma system or the stream-function used in fluids. This model equation represents a family of systems parametrized by the choice of $\mu(x, y)$ and $f(x, y)$. The HM-equation and Charney equation (see Example 2.2) corresponds to special cases in this family, which are shown in the examples below.

Example 1 $\left.\begin{array}{c}\mu^2=0\\ f=f_0+\beta y\end{array}\right\}\Longrightarrow (\triangle\psi)_t+\{\psi,\triangle\psi+f(y)\}=0$

This equation represents a 2-dimensional rotating incompressible fluid on a β-plane where f is the coriolis parameter (equal to twice the rotation frequency) and ψ is the streamfunction which represents the fluid velocity by $\mathbf{u}=(-\partial_y\psi,\partial_x\psi)$. The equation has no localized vortex solutions but it have dispersive linear waves called Rossby waves. The nonlinear interaction between these waves have recently been studied [9].

Example 2 $\left.\begin{array}{c}\mu^2=const.\\ f=f_0+\beta y\end{array}\right\}\Longrightarrow (\triangle\psi-\mu^2\psi)_t+\beta\psi_x+\{\psi,\triangle\psi\}=0$

We have obtained the Charney-equation, the fluid analog to HM-equation. Here μ is inverse of the Rossby radius, ψ represents the perturbation of the fluid surface and f is the coriolis parameter. This equation has, like the HM-equation, localized vortex solutions and it also contains Rossby waves.

Example 3 $\left.\begin{array}{c}\mu^2=T_0/T(y)\\ f=-\ln n_0(y)\end{array}\right\}\Longrightarrow$

$$\left(\triangle\psi-\frac{T_0}{T(y)}\psi\right)_t-\frac{1}{n_0(y)}\frac{dn_0}{dy}\psi_x+\frac{T_0}{T^2(y)}\frac{dT}{dy}\psi\psi_x+\{\psi,\triangle\psi\}=0$$

This is the HM-equation with an extra nonlinear term called the scalar nonlinearity (the third term), which have been studied extensively since it may describe large-scale monopolar vortices [8]. Here $T(y)$ is the electron temperature, $T_0=T(0)$, $n_0(y)$ is the plasma density and ψ is the perturbed electric potential.

These three examples shows that by consider the equation (1) we may obtain results for many types of systems. In next section we present a Hamiltonian structure of (1) which can be useful in the study of stability and interaction of waves.

3. The Hamiltonian structure

Many non-dissipative fluid systems, e.g. MHD equations and Vlasov systems, have shown to admit a Hamiltonian formulation. By a Hamiltonian formulation we mean that the dynamical equation can be put in the form

$$\partial_t\psi=X(\psi)=J_\psi H' \tag{2}$$

where H' is the functional derivative of the Hamiltonian H (H is usually the energy) and the operator J_ψ (the Poisson structure) is antisymmetric and satifies the Jacobi condition [6]. In addition to this we have to choose an inner product $\langle,\rangle$ that suites the problem.

The Hamiltonian formulation of the generalized HM-equation (1) can be written

$$\partial_t\psi=X(\psi)\equiv-\triangle_\mu^{-1}\{\psi,\triangle_\mu\psi+f\} \tag{3}$$

$$J_\psi a\equiv-\triangle_\mu^{-1}\{a,\triangle_\mu\psi+f\} \tag{4}$$

$$\langle a,b\rangle\equiv\int\left(\nabla a\cdot\nabla b+\mu^2ab\right)dxdy \tag{5}$$

$$H(\psi)\equiv\frac{1}{2}\langle\psi,\psi\rangle=\frac{1}{2}\int\left((\nabla\psi)^2+\mu^2\psi^2\right)dxdy \tag{6}$$

where a,b are arbitrary fieldvariables and the operator $\triangle_\mu^{-1}$ is defined by $\triangle_\mu^{-1}\triangle_\mu a=a$. The evolution equation (3) is written in terms of the potential ψ since it is easier to handle the boundary conditions for ψ than for $\triangle_\mu \psi$.

4. Applications

When we know that our equations are Hamiltonian, i.e. the equations can be put in the form (2), we have access to a couple of theorems [6] that can be used in the analysis of (3). If, for example, the equations have symmetries like $X(\psi(x+\delta, y, t)) = X(\psi(x, y, t))$ we have Noether-like theorem giving us a conserved quantity corresponding to momenta. Other continous symmetries would work as well. Another example is the theorems concerning three-wave interaction in arbitrary backgrounds. Here we can directly state the coupling equations with corresponding expressions for the the wave energy and coupling coefficients. From general theory we also know that the Manley-Rowe relations (describing the proportion of energy converted to each mode) are satisfied.

4.1 Wave interaction in presence of a vortex street

Wave interaction in presence of a sheared flow should be of importance in for example fusion plasma confinement since it has been shown that sheared flows can reduce anomalous transport losses (e.g. [5]). Below we presents results concerning a special case of a sheared flow, a vortex street. This type of flow has properties which should be interesting in confinement analysis: a vortex street is a combination of vortices and a simple sheared flow, it is a solution to the generalized HM-equation (1) and it is stable which means that it should be a natural state of the plasma (stability analysis of the vortex street and more details are to appear in a forthcoming paper).

In order to study traveling solutions we first translate our equations into a system travelling in x-direction with constant speed U, and then we consider a three wave interaction of drift modes. These drift modes, ψ^a, ψ^b and ψ^c, are solutions of the linearized equations and satisfies the resonance condition $\omega^a + \omega^b + \omega^c = 0$. In the presence of the stationary vortex street solution

$$\psi_e = 2A \ln \left[\frac{1}{b\sqrt{2}} (a \cosh(by) + (a^2-1)^{1/2} \cos(bx)) \right] - Uy \tag{7}$$

the coupling coefficient can be written

$$V^{abc} = \int_D \{c, Q_e\} \{a, \{\psi_e + Uy, b\}\} + i\omega^b \int_D \{c, Q_e\} \{a, b\} + (b \leftrightarrow c) \tag{8}$$

where a, b and c are generators defining the perturbation of the field as $\psi^a = J_{\psi_e} a$. The generalized potential vorticity, $Q_e \equiv \triangle_\mu \psi_e + f$, is in the case of a vortex street satisfying the relation $Q_e = A \exp(-(\psi_e + Uy)/A) - \mu^2 (\psi_e + Uy)$.

Due to its relativly simple structure the coupling coefficient (8) could be used in both numerical and analytical studies of, for example, interaction between turbulence and a sheared background flow like the vortex street.

5. References

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