

THREE DIMENSIONAL RESISTIVE MAGNETOHYDRODYNAMIC STABILITY

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This paper reports on the development of the three-dimensional resistive magnetohydrodynamic code for stellarator-like plasmas, SPECTOR-3D. The development was based on the previous resistive code for axisymmetric tokamak-like plasmas, SPECTOR [1]; however, since the problem is now 3D, it is an order of magnitude more complex. A number of codes have been developed to apply a similar model to toroidal plasmas (FAR [2], MARS [3], CASTOR [4], RESTOR [5] and SPECTOR [1]) and a modification of FAR has been developed for stellarators [6]. Although they differ in detail, the common feature of these codes is the use of a flux coordinate system based on the specific equilibrium. This is necessary, so that the effect of the dissipative terms can be accurately described in the vicinity of the rational q -surfaces where the perturbed magnetic field can vary dramatically. If the characteristic modes are represented by a double Fourier series in both the poloidal and the toroidal angles, toroidal and shaping effects lead to a coupling of the various poloidal modes even though the toroidal symmetry of the equilibrium projects out only a single toroidal mode. Two 3D ideal codes, TERPSICORE [7] and CAS3D [8] have been developed to study instabilities in ideal plasmas, although the use of a model kinetic energy term indicates the presence of unstable modes but not their actual growth rate. SPECTOR-3D aims to calculate the real growth rates for both ideal and resistive plasmas and to possibly explain new resistive phenomena which may occur in helical systems.

An axisymmetric toroidal plasma has to allow for coupling between the poloidal Fourier modes (labelled by m), but for a stellarator-like system there is also coupling between the toroidal modes (labelled by n) and the characteristic unstable modes need to be described in terms of a full double Fourier expansion. In addition the 3D algebraic analysis is significantly more complex. The code is designed to find the characteristic modes of the linear perturbations away from the plasma equilibrium, with an emphasis on the unstable modes.

The 3D plasma equilibrium code, VMEC, was interfaced as part of the input to SPECTOR-3D, incorporating a mapping to a new coordinate system - Boozer coordinates. This was necessary for both numerical analytic and continuity reasons. Boozer coordinates are necessary not only to maintain adequate continuity across the singular surfaces in the plasma but also to give a good description of the eigen-modes which vary rapidly in the direction orthogonal to the surfaces of constant flux. Boozer coordinates are a non-orthogonal set of coordinates and so there are considerable algebraic

complexities in reducing the vector MHD equations using these coordinates. As an example the driving term of the equation of motion contains over 150 terms which involve the metric tensor (all 9 components), the Jacobian and the magnetic field components. This makes the algebraic reduction of the equations difficult and dangerous, because of the high probability of algebraic or transcription errors. We developed a Mathematica program which carried out all of the non-orthogonal vector calculus, incorporating the double Fourier transform reduction, and as a bonus formulated the terms directly as FORTRAN statements which were incorporated into the code.

The reduction of the equations was done in terms of the 6 components of the vector potentials of the perturbed magnetic field and the velocity, which were in turn Fourier transformed in both the toroidal and poloidal angles. It is expected that typically more than 50 (m,n) modes will be required. The resulting set of about 300 second-order differential equations gives a block-tri-diagonal matrix after finite differencing. The growth rates are obtained in terms of the eigenvalues of this matrix. Typically a matrix involving about 300 sub-matrix blocks each of size 300x300 is obtained. The eigenvalue analysis is done by vector iteration. Because of the size of the problem the code was implemented on the VPP300 at the Australian National University using vector processing routines. The code has so far been checked against the results from axisymmetric toroidal codes ERATO (ideal) and SPECTOR (resistive) for low aspect ratio Solov'ev equilibria but will need extensive checking against 3D ideal MHD codes such as TERPSICHORE.

The equations used are the incompressible set of resistive MHD equations:

$$\frac{\partial \mathbf{v}}{\partial t} = (\nabla \times \mathbf{B}) \times \mathbf{b} + (\nabla \times \mathbf{b}) \times \mathbf{B} - \nabla p,$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \eta (\nabla \times \mathbf{b}).$$

with $\nabla \cdot \mathbf{b} = 0$ and $\nabla \cdot \mathbf{v} = 0$ (for incompressibility).

Using vector potentials for both the perturbed magnetic field and the velocity these are written in the form:

$$\frac{\partial (\nabla \times \nabla \times \mathbf{u})}{\partial t} = \nabla \times [(\nabla \times \mathbf{B}) \times (\nabla \times \mathbf{a}) + (\nabla \times \nabla \times \mathbf{a}) \times \mathbf{B}],$$

$$\frac{\partial \mathbf{a}}{\partial t} = (\nabla \times \mathbf{u}) \times \mathbf{B} - \eta (\nabla \times \nabla \times \mathbf{a}) + \nabla f,$$

where : $\mathbf{v} = \nabla \times \mathbf{u}$ and $\mathbf{b} = \nabla \times \mathbf{a}$, so that \mathbf{a} is the perturbed magnetic vector potential. If s labels the flux surfaces, the choice for the arbitrary gauge function, $f = \mathbf{B} \cdot \mathbf{u}$ leads to terms on the right hand side of the vector potential equation of the form $(m - nq(s))\mathbf{u}$. This shows explicitly the origin of the singularity in the ideal case ($\eta = 0$) for values of

the safety factor $q(s)$ equal to the ratio of the poloidal mode number m and the toroidal mode number n . This applies for both the toroidal and the 3d-helical case when the perturbations are expressed in terms of Fourier series in the toroidal and poloidal angle variables.

The equilibrium condition $\nabla P = \mathbf{J} \times \mathbf{B}$ needs to lead to nested flux surfaces with a Jacobian \mathcal{J} to define a flux-coordinate system (s, θ, ζ) . In this coordinate system

$$\mathbf{B} = B^\theta \mathcal{J} \nabla \zeta \times \nabla s + B^\zeta \mathcal{J} \nabla s \times \nabla \theta = \frac{f(s)}{\mathcal{J}} (\mathcal{J} \nabla \zeta \times \nabla s + q(s) \mathcal{J} \nabla s \times \nabla \theta)$$

which gives straight field lines with

$$\mathbf{B} \cdot \nabla \propto \frac{\partial}{\partial \theta} + q(s) \frac{\partial}{\partial \zeta}$$

We use the appropriate Jacobian for the Boozer coordinate system.

With

$$\mathbf{a} = a_s \nabla s + a_\theta \nabla \theta + a_\zeta \nabla \zeta,$$

the covariant components are expressed in a double Fourier series, e.g.

$$a_s(s, \theta, \zeta, t) = \sum_{m=M_1}^{M_2} \sum_{n=N_1}^{N_2} a_{smn}(s) \exp[im\theta - in\zeta - i\omega t]$$

For the axisymmetric (tokamak) case we use the poloidal sum only ($M = M_2 - M_1 + 1$ terms) but for the helical (stellarator) case both poloidal and toroidal sums ($N = N_2 - N_1 + 1$ terms) are needed. Not all modes need to be included so that a more convenient representation is

$$a_s(s, \theta, \zeta, t) = \sum_{k=0}^K a_{sm(k)n(k)}(s) \exp[im(k)\theta - in(k)\zeta - i\omega t].$$

Taking the double Fourier transform, the contravariant components of the Jacobian times the equation of motion

$$-i\omega \mathcal{J} (\nabla \times \nabla \times \mathbf{u}) = \mathcal{J} \nabla \times [\mathbf{J} \times (\nabla \times \mathbf{a}) + (\nabla \times \nabla \times \mathbf{a}) \times \mathbf{B}],$$

become

$$\begin{aligned} -i\omega \sum_m \sum_n \sum_\beta [R_{1\alpha\beta}(m', m, n', n) u_{\beta mn}(s) + R_{2\alpha\beta}(m', m, n', n) \frac{\partial u_{\beta mn}}{\partial s} \\ + \frac{\partial}{\partial s} (R_{3\alpha\beta}(m', m, n', n) u_{\beta mn}) + \frac{\partial}{\partial s} (R_{4\alpha\beta}(m', m, n', n) \frac{\partial u_{\beta mn}}{\partial s})] \\ = \sum_m \sum_n \sum_\beta [P_{1\alpha\beta}(m', m, n', n) u_{\beta mn}(s) + P_{2\alpha\beta}(m', m, n', n) \frac{\partial a_{\beta mn}}{\partial s} \\ + \frac{\partial}{\partial s} (P_{3\alpha\beta}(m', m, n', n) u_{\beta mn}) + \frac{\partial}{\partial s} (P_{4\alpha\beta}(m', m, n', n) \frac{\partial a_{\beta mn}}{\partial s})] \end{aligned}$$

where $(\alpha, \beta) = (s, \theta, \zeta)$ and $R_{1\alpha\beta}(m', m, n', n)$ etc are double Fourier coefficients of the terms arising from the convolution. A similar structure arises from the Fourier decomposition of Amperes law. In all a total of 105 functions have to be transformed to obtain the coefficients. The algebra is challenging and involves a 3D non-orthogonal coordinate vector analysis. MATHEMATICA was used to do this. Inverse iteration is used to search for the eigenvalues of the system.

SPECTOR-3D has been implemented on the Fujitsu VPP300 at the Australian National University Supercomputer Facility. The code consists of 5 distinct phases.

1. Solution of MHD equilibrium using VMEC

The time taken depends on the number of mode pairs required.

2. Mapping the coordinate system to Boozer coordinates

This calculates the input data needed *viz.* the contravariant components of the equilibrium magnetic field, the Jacobian and the covariant components of the metric tensor. This is relatively fast.

3. Set up the block tri-diagonal matrices

The most time consuming part of this is the calculation of the double Fourier transforms. Typically this takes 0.5 seconds per surface for $(32 \theta \times 32 \zeta)$ divisions and 1.8 seconds per surface for (64×64) for , say, 100 surfaces = **180** seconds. This lends itself to parallelisation

4. Invert the block tri-diagonal matrix

This depends on the number of surfaces (\mathcal{N}) and the number of modes (K). Typically the time taken is $5 \times 10^{-4} \mathcal{N} K^2 = \mathbf{125}$ seconds for 100 surfaces and 50 modes. It is difficult to parallelise this phase.

5. Inverse Iteration

The number of iterations depends on the initial guess. One usually hopes to converge by 20 steps. Typically the time taken is $2 \times 10^{-5} \mathcal{N} K^2 = 5$ seconds per iteration for 100 surfaces and 50 modes giving about **100** seconds for 20 iterations. This needs to be sequential but the matrix multiplications can be vectorised.

This work has been supported by the Australian Research Grants Committee. We thank Dr S. P. Hirshman for the use of the VMEC equilibrium code and acknowledge contributions from R. L. Dewar and G. Y. Fu.

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