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Parallel Flows and Plasma Equilibria in Dipolar Magnetic Configurations

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Abstract

The effects of parallel plasma flow on plasma equilibrium in dipolar magnetic configurations (MC) are considered by applying a separable form of magnetic flux function. Both analytical and numerical solutions of the plasma equilibria for sub-Alfvenic plasma flow are presented. However, no physically meaningful solutions with super-Alfvenic plasma flow are found.

I. Introduction. Plasma equilibria in a dipolar MC and the effects of the plasma flows on these equilibria are of interest for both laboratory experiments [1-4] and astrophysical applications [5,6]. The effects of parallel plasma flows on plasma equilibrium in dipolar MC are the topic for this study. From the ideal MHD equations assuming toroidal symmetry we can derive [7] the following equations for the plasma equilibrium in a dipolar MC.

$$\rho\vec{v} = \vec{B}\Phi(\psi) \quad (1) \qquad\qquad \vec{B}\cdot\left(\frac{\rho\nabla v^2}{2} + \nabla P\right) = 0 \qquad (2)$$

$$\left\{\nabla\cdot\left[\left(1-\frac{\Phi^2}{\rho}\right)\frac{1}{R^2}\nabla\psi\right] + \frac{B^2}{2\rho}\frac{d\Phi^2}{d\psi}\right\}|\nabla\psi|^2 + \left(\frac{\rho\nabla v^2}{2} + \nabla P\right)\nabla\psi = 0\,, \qquad (3)$$

where ρ, $\vec{v}$, and P are the plasma mass density, velocity, and pressure; ψ is the magnetic flux function; $\vec{B} = \nabla\psi\times\nabla\varphi$ is the magnetic field; $\Phi(\psi)$ is the arbitrary function of ψ; and R is the distance from the major axis. To reduce Eq. (1)-(3) we will follow [8] and introduce a separable form for the flux function ψ written in spherical coordinates ($r,\varphi,\mu = \cos\vartheta$), i.e. the *anzatz*

$$\psi(\mu, r) = \psi_0 h(\mu)(r_0 / r)^{\alpha}\ , \qquad (4)$$

where α is an adjustable parameter, $h(\mu)$ is an unknown function of μ alone, and ψ_0 and r_0 are normalization constants $h(\mu)$ and α play the roles of the eigenfunction and eigenvalue of the nonlinear Grad-Shafranov equation. To apply *anzatz* (4) we need (see Eq. 3) to assume

$$(\Phi(\varphi))^2/\rho(\vec{r}) \equiv (v/V_A)^2 \equiv W(\mu)\,, \tag{5}$$

where V_A is the Alfven velocity and $W(\mu)$ is some function of μ alone. Substituting Eq. (5) in (2) and (3) and assuming that P=0 we find that product BW supposed to be a function of ψ. Since W is a function of μ alone, using the *anzatz* (4), which corresponds to $B^2 \propto \left(\alpha^2 h^2/(1-\mu^2)+(dh/d\mu)^2\right)^{-2(\alpha+2)}$, we find the expression for $W(\mu)$,

$$W(\mu) = W_0 \frac{h^{(\alpha+2)/\alpha}}{\left(h^2/(1-\mu^2)+\alpha^{-2}(dh/d\mu)^2\right)^{1/2}}\,, \tag{6}$$

where $W_0 \geq 0$ is a constant, as well as the equation for the function $h(\mu)$

$$\frac{d}{d\mu}\left((1-W)\frac{dh}{d\mu}\right)+\alpha(\alpha+1)\frac{(1-W)h}{1-\mu^2}+\alpha(\alpha+2)\frac{W_0^2}{W}h^{(\alpha+4)/\alpha}=0\,, \tag{7}$$

which describes equilibrium of pressureless plasma with parallel flow in dipolar MC. As the boundary conditions we can take

$$h(\mu^2 \to 1) \propto 1-\mu^2,\ h(\mu=0)=1, \text{ and } (dh/d\mu)\big|_{\mu=0}=0\,, \tag{8}$$

which implies, finite magnetic field at the major axis, normalization of the constant ψ_0 (4), and “up-down” symmetry of the MC. Thus the 2-nd order Eq. (7) having 3 boundary conditions is over-determined and it’s solution is only possible for some special relation between the constants W_0 and α, $\alpha=\alpha(W_0)$. From Eq. (7) we need to find both $\alpha=\alpha(W_0)$ and $h(\mu)$, corresponding to this α. We will consider only positive α. Then at a given flux surface ψ we have $r \propto h(\mu)^{1/\alpha}\big|_{\mu\to\pm1} \to 0$ and the magnetic flux is reminiscent to the MC of a point dipole.

II. Solution. ***A.The case*** $W_0<1$. For $W_0 \ll 1$ we find the following dependence $\alpha=\alpha(W_0)$

$$\alpha-1 \approx -\frac{W_0}{2}\int_0^1 d\mu \frac{\left(1-\mu^2\right)^3\left(1+15\mu^2\right)}{\left(1+3\mu^2\right)^{1/2}} \approx -0.49\times W_0\,. \tag{9}$$

Eigenfunction $h(\mu)$ corresponding to the above α we find from the following expression

$$\frac{1-f(\mu)}{W_0} \approx \hat{f}(\mu) \equiv \int_0^{\mu} d\mu' \left\{ \frac{2\left(1-\mu'^2\right)^2}{\left(1+3\mu'^2\right)^{1/2}} + \frac{1}{2}\left[(2+\mu')\int_0^{\mu'} d\mu'' \frac{\left(1-\mu''^2\right)^2\left(1+15\mu''^2\right)}{\left(1+3\mu''^2\right)^{1/2}} \right.\right.$$

$$\left.\left. -\mu'\left(3-\mu'^2\right)\int_{\mu'}^{1} d\mu'' \left(\frac{1-\mu''^2}{1-\mu'^2}\right)^2 \frac{\left(1+15\mu''^2\right)}{\left(1+3\mu''^2\right)^{1/2}} \right]\right\}. \qquad (10)$$

here $f(\mu) = h(\mu)/(1-\mu^2)$. The function $\hat{f}(\mu)$ found numerically from (10) is shown in Fig.1. For $W_0 \tilde{<} 1$ the equation (7) with the boundary conditions (8) was solved numerically. The dependence $\alpha(W_0)$ found from these calculations is shown in Fig.2. Note that for $W_0 < 1$ the eigenvalue $\alpha(W_0)$ found from the numerical solution agrees well with analytic estimate (9). ***B. The case*** $W_0 > 1$**.** This case could correspond to the super-Alfvenic plasma flow if the boundary conditions (8) would be applicable. But it can be shown analytically that $W_0 > 1$ is not compatible with the boundary conditions (8), which describe well-behaved dipolar configuration with finite magnetic field. However we can relax the boundary conditions (8) and consider equilibria with an azimuthal current sheet located at $\mu = 0$. In this case the dipolar magnetic field line changes direction when it goes through the current sheet causing $(dh/d\mu)|_{\mu=\pm 0}$ to be finite and discontinuous. This extra freedom allows both $h(\mu = 0) = 1$ and $h(\mu^2 \to 1) \propto 1-\mu^2$ to be satisfied. Therefore, when azimuthal current sheet is allowed at $\mu = 0$ the boundary conditions for Eq.(7) become

$$h(\mu = 0) = 1, (p(\mu = \pm 0))^2 = p_0^2 \equiv W_0^2 - 1 \text{ and } h(\mu^2 \to 1) \propto 1-\mu^2, \qquad (11)$$

Notice, that the plasma flow velocity at $\mu = 0$ just reaches the Alfven velocity even when $W_0 > 1$**.** For $W_0 >> 1$ the solution of Eq.(7) with boundary conditions (11) can be found analytically. From (7) we see that for $p^2 >> 1$ we have $dp/d\mu \sim O(1)$. Since μ varies from 0 to 1 we can neglect the variation of p and assume that $p^2 \approx p_0^2 >> 1$. Then from (7) we find that to satisfy (11) we should have $\alpha \approx 1/W_0$ and $h(\mu) \approx 1-|\mu|$. Notice, that flux surfaces corresponding to this solution are squeezed about equatorial plane. We also solve Eqs. (7),(11) numerically to determine the function $\alpha(W_0)$ which is shown in Fig.3. We find that

α varies continuously as W_0 passes through unity and both $\alpha(W_0)$ and $h(\mu)$ approach the limit obtained analytically at large W_0.

III. Conclusions. We find both analytical and numerical solutions of the equation describing plasma equilibrium in dipolar MC for sub-Alfvénic plasma flow and no physically meaningful solutions with super-Alfvenic plasma flow. We conclude that there is no steady-state axisymmetric equilibrium permitting super-Alfvénic parallel plasma flow for the class of separable dipolar magnetic configurations considered.

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Figures

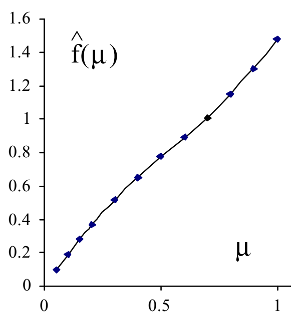

Fig.1

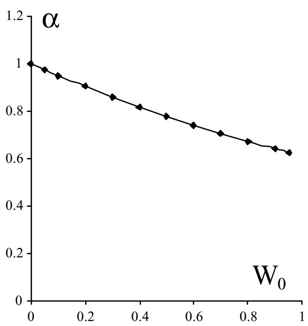

Fig.2

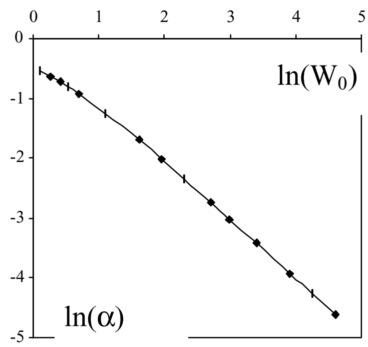

Fig.3

Fig. 1 The function $\hat{f}(\mu)$ found from numerical integration of expression (10).
Fig. 2. $\alpha(W_0)$ found from numerical solution of Eq. (7) and (8) for $W_0 < 1$.
Fig. 3. $\alpha(W_0)$ found from numerical solution of Eq.(7) and (11) for $W_0 > 1$.