

MHD Stability of Axisymmetric Plasmas in Closed Line Magnetic Fields

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Abstract

The stability of axisymmetric plasma confined by a poloidal magnetic field with closed lines is investigated using MHD equations with anisotropic resistivity. It is shown that, in up-down symmetric systems, a resistive instability with a growth rate proportional to the cubic root of the resistivity is necessarily present at the ideal stability boundary for up-down antisymmetric modes at finite beta. Both the ideal and resistive stability of a Z pinch and a point dipole equilibria are studied in detail.

1. Introduction

Plasmas confined by axisymmetric closed line poloidal magnetic fields are common in both nature (examples are stellar and planetary ionospheres) and the laboratory (Z pinches, field reversed configurations, multipoles and so on). In such systems, plasma and magnetic field compression due to closed field lines or large trapped particle populations can counteract unfavorable magnetic field line curvature, providing favorable stability properties, as demonstrated by ideal MHD calculations [1, 2] for a magnetic dipole.

In this paper, we concentrate on the effects of plasma resistivity on the stability of such systems. Unlike the situation in tokamaks and stellarators where resistive modes are strongly radially localized about isolated rational flux surfaces, all flux surfaces are rational in closed field line systems so that strong radial localization of resistive modes does not occur.

We begin by formulating and discussing the necessary equations in Sec. 2 and then use them in Secs. 3 and 4 to study resistive stability of a Z pinch and the point dipole equilibrium [3], respectively.

2. Equations

Using standard linearized MHD equations with anisotropic resistivity and considering the (most unstable [4]) limit of large azimuthal mode numbers, $n \gg 1$, it is possible to derive the following system of equations for the radial component ξ_ψ of the plasma

displacement $\boldsymbol{\xi}$ and a quantity $W \equiv 4\pi\Gamma p (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})$,

$$\mathbf{B} \cdot \boldsymbol{\nabla} \left[\frac{(\mathbf{B} \cdot \boldsymbol{\nabla} \xi_\psi)}{R^2 B^2 (1 + c^2 n^2 \eta_{\parallel} / 4\pi R^2 \gamma)} \right] + 2 \left(\frac{\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \psi}{R^2 B^2} \right) \left(4\pi \frac{dp}{d\psi} \xi_\psi + W \right) = \frac{4\pi \rho \gamma^2}{R^2 B^2} \xi_\psi, \quad (1)$$

$$\mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{\mathbf{B} \cdot \boldsymbol{\nabla} W}{B^2} \right) - 2 \frac{c^2 n^2 \eta_{\parallel}}{\gamma} \frac{dp}{d\psi} \left(\frac{\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \psi}{R^2 B^2} \right) \left(4\pi \frac{dp}{d\psi} \xi_\psi + W \right) = \quad (2)$$

$$\frac{\rho \gamma^2}{\Gamma p} \left[1 + \frac{4\pi \Gamma p}{B^2} \left(1 + \frac{c^2 n^2 \eta_{\perp}}{4\pi R^2 \gamma} \right) \right] W + 4\pi \rho \gamma^2 \left[2 \left(\frac{\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \psi}{R^2 B^2} \right) + \frac{c^2 n^2 (\eta_{\perp} - \eta_{\parallel})}{\gamma R^2 B^2} \frac{dp}{d\psi} \right] \xi_\psi,$$

where the magnetic field is $\mathbf{B} = \boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} \zeta$ with ψ the poloidal magnetic flux and ζ the toroidal angle, $\boldsymbol{\kappa} \equiv \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \hat{\mathbf{n}}$ is the magnetic field curvature with $\hat{\mathbf{n}} \equiv \mathbf{B}/|\mathbf{B}|$, R is the cylindrical radial coordinate, p and ρ are plasma pressure and density, respectively, Γ is the ratio of specific heats at constant pressure and volume, η_{\parallel} and η_{\perp} are the parallel and perpendicular resistivities, γ is the mode growth rate, and c is the speed of light.

The system of Eqs. (1) and (2) can be considerably simplified for ideal modes near marginality leading to the well-known ballooning equation for shear Alfvén modes [2, 4],

$$\mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{\mathbf{B} \cdot \boldsymbol{\nabla} \xi_\psi}{R^2 B^2} \right) + 4\pi \frac{2(\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} p) - \rho \gamma^2}{R^2 B^2} \xi_\psi = 16\pi \Gamma p \left(\frac{\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \psi}{R^2 B^2} \right) \frac{\langle \xi_\psi (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \psi) / R^2 B^2 \rangle_{\theta}}{1 + 4\pi \Gamma p \langle B^{-2} \rangle_{\theta}},$$

where $\langle \dots \rangle_{\theta} \equiv V^{-1} \oint [(\dots) d\theta / \mathbf{B} \cdot \boldsymbol{\nabla} \theta]$, $V \equiv \oint [d\theta / \mathbf{B} \cdot \boldsymbol{\nabla} \theta]$ and θ is a poloidal angle.

As $\eta_{\parallel, \perp}$ is normally small the terms $(c^2 n^2 \eta / 4\pi R^2 \gamma) \equiv (1/\tau_{res} \gamma)$ can be important only near ideal stability boundaries where γ is small. If we consider for the moment both axially *and* up-down symmetric systems and modes with $\gamma \sim \tau_A^{-2/3} \tau_{res}^{-1/3} \propto \eta^{1/3}$, where $\tau_A \equiv \sqrt{4\pi \rho R^2 / B^2}$, so that $(1/\tau_{res} \gamma) \sim (\tau_A / \tau_{res})^{2/3} \equiv \delta \ll 1$, then assuming $\beta \sim 1$ and expanding ξ_ψ and W in powers of δ , we can solve Eqs. (1) and (2) order by order separately for up-down symmetric (even) and antisymmetric (odd) modes. We find that when ideal marginality is accessible there is always an unstable odd resistive mode at the ideal odd mode stability boundary with a growth rate

$$\gamma = [(c^2 n^2 \eta_{\parallel} / \rho) (dp/d\psi)^2 F^{-1}]^{1/3} \propto \eta_{\parallel}^{1/3} \quad (3)$$

where F is a non-negative equilibrium quantity. There is not an unstable even mode at the ideal even mode stability boundary. The latter result is not unexpected as a mode with $\gamma \propto \eta^{1/3}$ comes about due to a decrease of stabilizing field line bending energy by parallel resistivity and the lowest even ideal mode is a flute mode at its stability boundary so that the corresponding field line bending energy is zero. For a Z pinch $F = 0$ and Eq. (3) should not be used. Instead, it can be shown that there is no resistive modes with $\gamma \propto \eta^{1/3}$ at $\beta \sim 1$ in this special case.

3. Resistive Stability of a Z pinch Equilibrium

For a Z pinch $(\mathbf{B} \cdot \nabla) = i(mB/R)$ with m a poloidal mode number. Equations (1) and (2) then lead to the fourth order polynomial dispersion relation

$$\begin{aligned} \tilde{\gamma}^2 \{ [\tilde{\gamma}^2 + m^2 \tilde{\gamma} / (\tilde{\gamma} + \tilde{\eta}_{\parallel})] [1 + \Delta (1 + \tilde{\eta}_{\perp} / \tilde{\gamma})] + \Delta (4 + m^2) - \alpha (1 + \Delta) \} \\ + \Delta m^2 (m^2 - \alpha) \tilde{\gamma} / (\tilde{\gamma} + \tilde{\eta}_{\parallel}) = 0, \end{aligned} \quad (4)$$

with $\tilde{\gamma} \equiv \gamma \tau_A$, $\Delta \equiv \Gamma \beta / 2$, $\beta \equiv 8\pi p / B^2$, $\alpha \equiv -\beta (d \ln p / d \ln R)$, and $\tilde{\eta}_{\parallel, \perp} \equiv (c^2 n^2 \eta_{\parallel, \perp} / 4\pi R^2) \tau_A$. Modes with different m do not interact so that Eq. (4) can be solved separately for $m = 0$ or *interchange* modes and $m > 0$ modes.

The interchange modes are represented in this case by two shear Alfvén modes that are unstable when $\alpha > 4\Delta / (1 + \Delta)$ or equivalently when $-d \ln p / d \ln V > \Gamma$ and are damped by perpendicular resistivity otherwise, the stability condition being identical to the usual interchange stability condition [5]. Parallel resistivity plays no role in the stability of $m = 0$ modes.

Stability analysis of $m > 0$ modes is more involved. In the ideal case four modes are present which are strongly coupled at $\beta \sim 1$, but uncouple at $\beta \ll 1$ giving two stable sound waves with $\tilde{\gamma}_s = \pm i m \sqrt{\Delta}$ and two shear Alfvén modes with $\tilde{\gamma}_A = \pm i \sqrt{m^2 - \alpha}$. At arbitrary β one of the ideal modes is unstable when $\alpha > m^2$, but all four of them are stable otherwise. Retaining resistive terms with $0 \leq \tilde{\eta}_{\parallel}, \tilde{\eta}_{\perp} \ll 1$ leads to small corrections to $\tilde{\gamma}$ at finite β which are proportional to resistivity and can be either stabilizing or destabilizing. However, the pressure gradients required for these corrections to be destabilizing are strong enough to necessarily destabilize $m = 0$ modes.

4. Resistive Stability of the Point Dipole Equilibrium

Next, we consider the MHD stability of the separable point dipole equilibrium [3] by solving Eqs. (1) and (2) using the appropriate geometry and boundary conditions. In the ideal case hierarchies of even and odd modes formed by interacting sound waves and shear Alfvén modes are obtained. These modes are stable at arbitrary β when $\Gamma = 5/3$. As an example, a plot of $\tilde{\omega}^2 = -\tilde{\gamma}^2$ vs. β for the even modes is given in Fig. 1. Retaining resistivity results in small growth or decay rates of the ideal modes which always scale linearly with resistivity. These rates are shown in Fig. 2 for the lowest even mode for the situations when $(\tilde{\eta}_{\perp}, \tilde{\eta}_{\parallel}) = (0.1, 0)$, $(\tilde{\eta}_{\perp}, \tilde{\eta}_{\parallel}) = (0, 0.1)$ and $(\tilde{\eta}_{\perp}, \tilde{\eta}_{\parallel}) = (0.196, 0.1)$ by dashed, dashed-dotted and dotted lines, respectively. The fact that a resistive instability with $\tilde{\gamma} \propto \tilde{\eta}_{\parallel}^{1/3}$ has not been found here is consistent with the analysis of Sec. 2, as odd

modes are always ideally stable for the point dipole equilibrium [3].

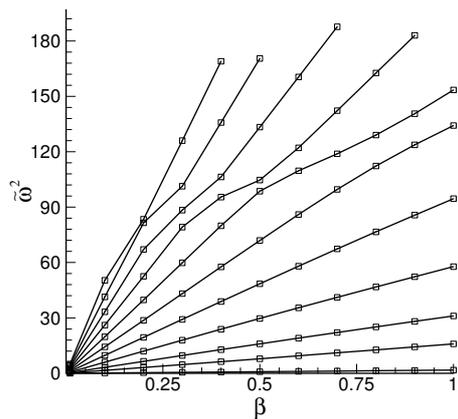


Figure 1.

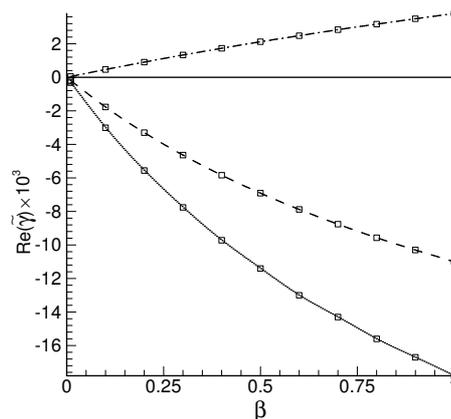


Figure 2.

5. Conclusions

The stability of plasma in axisymmetric closed line poloidal magnetic field has been investigated using an MHD description with anisotropic resistivity. The system of two coupled Eqs. (1) and (2) describing this stability has been presented. It has been shown that in both axially *and* up-down symmetric systems like a magnetic dipole (but not a Z pinch) a resistive instability with a growth rate $\propto (\text{resistivity})^{1/3}$ always exists at the ideal stability boundary for up-down antisymmetric modes. The stability of Z pinch and point dipole [3] equilibria have been studied in detail. The point dipole equilibrium has been found to be ideally stable at arbitrary β when $\Gamma = 5/3$, while the ideal stability criteria for a Z pinch equilibrium has been found to be in agreement with the original work by Kadomtsev [6]. Resistive instabilities have been found to be unimportant for a Z pinch, but may be capable of destabilizing the point dipole. The growth rate of such instabilities is $\gamma \propto n^2\eta$, but is never $\gamma \propto \eta^{1/3}$ as the point dipole equilibrium is always robustly stable with respect to ideal up-down antisymmetric modes.

Acknowledgements

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