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Electromagnetic Wave Energy Transport in a Weakly Inhomogeneous and Slightly Dissipative Medium with Embedded Sources

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The equation governing the energy transport for wave packets propagating in weakly inhomogeneous and slightly dissipative media with embedded sources is obtained on the basis of the generalized Fourier integral representation of the wave field.

Introduction. As is well known, on dealing with electromagnetic waves propagating in a homogeneous medium, the wave field is conveniently represented by means of the Fourier transform which allows to describe a generic function of space and time as a superposition of plane waves, $g_0(\mathbf{k},\omega,\mathbf{r},t) = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Since g_0 are a complete set of functions, $\int d^3k d\omega g_0(\mathbf{k},\omega,\mathbf{r},t) g_0^*(\mathbf{k},\omega,\mathbf{r}',t') = (2\pi)^4\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, it follows that any wave packet can be described by using the Fourier integral representation of the field and the effect of a generic source current, $\mathbf{j}(\mathbf{r},t)$, can be accounted for in terms of its Fourier amplitude $\mathbf{j}(\mathbf{k},\omega)$. Furthermore, the plane waves are each other independent due to their orthogonality, $\int d^3r dt g_0(\mathbf{k},\omega,\mathbf{r},t) g_0^*(\mathbf{k}',\omega',\mathbf{r},t) = (2\pi)^4\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')$. For the case of propagation in weakly inhomogeneous media, the Fourier transform method, which no longer provides the solution of Maxwell's equations, is replaced by the geometrical optics (GO) eikonal ansatz for the wave electric field [1, 2], $\mathbf{E}(\mathbf{r},t) = \Re e\left\{\mathbf{A}(\mathbf{r},t)e^{i\psi(\mathbf{r},t)}\right\}$, where the amplitude $\mathbf{A}$ as well as $\frac{\partial\psi}{\partial\mathbf{r}}(=\mathbf{k}$, the (local) wave vector) and $\frac{\partial\psi}{\partial t}(=-\omega$, the (local) wave frequency) are slowly varying functions of both $\mathbf{r}$ and t. This single quasi-plane wave representation needs to be extended to describe both the propagation of a generic wave packet and the effects of external source currents. Such an extension can be accomplished through the generalized Fourier integral (GFI) representation [3, 4],

$$\mathbf{E}(\mathbf{r},t) = \Re e \int \frac{d^3\mathrm{a}d\alpha}{(2\pi)^4}\mathbf{E}(\mathbf{a},\alpha,\mathbf{r},t)\; e^{i\psi(\mathbf{a},\alpha,\mathbf{r},t)}, \tag{1}$$

where the Fourier amplitude $\mathbf{E}(\mathbf{k},\omega)$ and the plane wave phase, $\mathbf{k}\cdot\mathbf{r}-\omega t$, are replaced by $\mathbf{E}(\mathbf{a},\alpha,\mathbf{r},t)$ and the eikonal $\psi(\mathbf{a},\alpha,\mathbf{r},t)$, respectively, labelled by the parameters $\mathbf{a}$ and α, the local wave vector and frequency being defined the same as in the eikonal theory. The GFI representation (1) amounts to a superposition of complete, orthogonal quasi-plane waves provided that **(i)** the amplitude, the wave vector and the frequency are slowly varying functions in both space and time, i.e.,

$$\frac{1}{kf}\left|\frac{\partial f}{\partial \mathbf{r}}\right| \sim \frac{1}{\omega f}\left|\frac{\partial f}{\partial t}\right| \sim \delta \ll 1, \quad \text{with } f \text{ denoting any derivative of } \mathbf{A},\ \mathbf{k},\ \omega \text{ and}$$

δ being a small parameter, which for space- and time-varying media is identified with $\frac{2\pi}{kL} \sim \frac{2\pi}{\omega T}$, ($L$ and T being the space and time scales of the medium variations);

(ii) the eikonal funtion ψ defines a coordinate transformation, $\mathbf{k} = \frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{a}, \alpha, \mathbf{r}, t)$, $\omega = -\frac{\partial}{\partial t}\psi(\mathbf{k}, \omega, \mathbf{r}, t)$ for each fixed $(\mathbf{r}, t)$, the Jacobian determinant of which is $U = \left\| \frac{\partial(\mathbf{k},\omega)}{\partial(\mathbf{a},\alpha)} \right\|_{(\mathbf{r},t)} \neq 0$. In particular, condition **(i)** implies that each harmonic component of the wave packet (1) is a quasi-plane wave; condition **(ii)**, on the other hand, yields that the set of functions $g(\mathbf{a}, \alpha, \mathbf{r}, t) = U^{\frac{1}{2}} e^{i\psi}$ is asymptotically complete and orthogonal [3],

$$\int d^3 a d\alpha g(\mathbf{a}, \alpha, \mathbf{r}, t) g^*(\mathbf{a}, \alpha, \mathbf{r}', t') = (2\pi)^4 \delta(\mathbf{r} - \mathbf{r}')\delta(t - t') + O(\delta), \tag{2}$$

$$\int d^3 r dt g(\mathbf{a}, \alpha, \mathbf{r}, t) g^*(\mathbf{a}', \alpha', \mathbf{r}, t) = (2\pi)^4 \delta(\mathbf{a} - \mathbf{a}')\delta(\alpha - \alpha') + O(\delta), \tag{3}$$

the two relevant properties of the plane waves $g_0(\mathbf{k}, \omega, \mathbf{r}, t)$ being thus preserved to lowest order in δ.

The transport of the wave energy density in phase space. In the limit of weakly non-Hermitian media ($|\underline{\underline{\varepsilon}}_a| \sim O(\delta)|\underline{\underline{\varepsilon}}_h|$, $\underline{\underline{\varepsilon}}_{(a)h}$ being the (anti)-Hermitian part of the dielectric tensor), the equation for the GFI amplitude of the wave electric field, $\mathbf{E} = \mathbf{E}(\mathbf{a}, \alpha, \mathbf{r}, t)$, is

$$\int \frac{d^3 a d\alpha}{(2\pi)^4} e^{i\psi(\mathbf{a},\alpha,\mathbf{r},t)} \omega^2 \left\{ \underline{\underline{\Lambda}} \cdot \mathbf{E} + \frac{4\pi i}{\omega}\mathbf{j} + i\left(\underline{\underline{\varepsilon}}_a \cdot \mathbf{E} + \hat{\mathsf{F}}[\mathbf{E}]\right) + O(\delta^{\frac{3}{2}}) \right\} = 0, \tag{4}$$

which, multiplying by $e^{-i\psi(\mathbf{a}',\alpha',\mathbf{r},t)}$ and integrating on space and time, amounts to

$$\underline{\underline{\Lambda}} \cdot \mathbf{E} = -\frac{4\pi i}{\omega}\mathbf{j} - i\left(\underline{\underline{\varepsilon}}_a \cdot \mathbf{E} + \hat{\mathsf{F}}[\mathbf{E}]\right) - \frac{i}{\omega^2}\left[\frac{\partial}{\partial \omega}\frac{\partial}{\partial t} - \frac{\partial}{\partial \mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{r}}\right]\omega^2 \underline{\underline{\Lambda}} \cdot \mathbf{E} + +O(\delta^{\frac{3}{2}}). \tag{5}$$

In Eqs.(4) and (5), $\underline{\underline{\Lambda}}(\mathbf{k}, \omega, \mathbf{r}, t) = \frac{c^2 k^2}{\omega^2}\left(\hat{\mathbf{k}}\hat{\mathbf{k}} - \underline{\underline{I}}\right) + \underline{\underline{\varepsilon}}_h$ is the local (Hermitian) dispersion tensor; $\hat{\mathsf{F}}[\mathbf{E}](= O(\delta))$ is a differential operator accounting for the weak inhomogeneity of the medium; the (slowly varying) GFI amplitude $\mathbf{j} \equiv \mathbf{j}(\mathbf{a}, \alpha, \mathbf{r}, t)$ of a given source current, $\mathbf{j}(\mathbf{r}, t)$, is obtained by making use of the completeness property (2) into the identity $\mathbf{j}(\mathbf{r}, t) = \int d^3 r' dt' \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\mathbf{j}(\mathbf{r}', t')$; furthermore, it is assumed that $|\mathbf{j}| = O(\delta^{\frac{1}{2}})$ so that the emitted power $P \propto |\mathbf{j}|^2$ is of $O(\delta)$. The last explicit term on the r.h.s. of (5) is due to higher order corrections in the orthogonality condition (3). On dealing with the wave energy density transport we consider the (ill-defined) quadratic form,

$$\underline{\underline{W}}(\mathbf{a}, \alpha, \mathbf{r}, t) \equiv \mathbf{E}(\mathbf{a}, \alpha, \mathbf{r}, t)\mathbf{E}^*(\mathbf{a}, \alpha, \mathbf{r}, t). \tag{6}$$

Making use into (5) of an asymptotic expansion in half-integer powers of $\delta(\ll 1)$ for the electric field, $\mathbf{E} = \sum_n \mathbf{E}^{(\frac{n}{2})}$ with $|\mathbf{E}^{(\frac{n}{2})}| = O(\delta^{\frac{n}{2}})|\mathbf{E}^{(0)}|$, yields the lowest three order equations for $\underline{\underline{W}}$,

$$\underline{\underline{\Lambda}} \cdot \underline{\underline{W}}^{(0)} = 0, \tag{7}$$

$$\underline{\underline{\Lambda}} \cdot \underline{\underline{W}}^{(\frac{1}{2})} = -\frac{4\pi i}{\omega}\mathbf{j}\mathbf{E}^{(0)}, \tag{8}$$

$$\underline{\underline{\Lambda}} \cdot \underline{\underline{W}}^{(1)} = \left(\frac{4\pi}{\omega}\right)^2 \mathbf{j}\left(\underline{\underline{\Lambda}}^{-1} \cdot \mathbf{j}\right)^* - i\underline{\underline{\varepsilon}}_a \cdot \underline{\underline{W}}^{(0)} - i\hat{\mathsf{F}}[\mathbf{E}^{(0)}]\mathbf{E}^{(0)*}. \tag{9}$$

With reference to (9), it is worth noting that the tensor $\underline{\underline{\Lambda}}^{-1}(\equiv \sum_i \left(\mathcal{P}\frac{1}{\Lambda_i} - i\pi\delta(\Lambda_i)\right)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_i^*$, $\hat{\mathbf{e}}_i$ denoting the (complex) eigenvectors of $\underline{\underline{\Lambda}}$, i.e., the polarization unit vectors, corresponding to the (real) eigenvalue Λ_i) has a singular imaginary part. The lowest-order equation (7) implies that $\underline{\underline{\Lambda}}$ and $\underline{\underline{W}}^{(0)}$ are diagonal on the same basis, i.e., in particular, $\underline{\underline{W}}^{(0)} = \sum_i W_i^{(0)}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_i^*$, so that in this basis Eq.(7) reduces to a set of three independent scalar equations $\Lambda_i W_i^{(0)} = 0$, with the result that $W_i^{(0)} = 0$ everywhere except on the set of dispersion surfaces determined by $\Lambda_i = 0$ in the $(\mathbf{k}, \omega, \mathbf{r}, t)$ space, each one corresponding to a specific propagating mode [5]. Thus the solution $\underline{\underline{W}}^{(0)}$ of (7) relevant to the mode σ is proportional to $\delta(H^\sigma)\hat{\mathbf{e}}^\sigma\hat{\mathbf{e}}^{\sigma *}$, H^σ being an arbitrary function such that $H^\sigma = 0$ yields the dispersion surface of the considered mode. On the other hand, $H^\sigma = 0$ is a first order partial differential equation for the eikonal ψ^σ, the characteritics of which are the (geometrical optics) rays in the eight-dimensional phase space $(\mathbf{k}, \omega, \mathbf{r}, t)$, and are formally equivalent to a four-dimensional dynamical system with conserved Hamiltonian H^σ. The dispersion equation amounts to setting the energy of the equivalent dynamical system to zero, hence it is a specific case of the corresponding Hamilton-Jacobi equation $H^\sigma(\mathbf{k}, \omega, \mathbf{r}, t) = \alpha$, a complete solution of which can be achieved provided that three constants of motion $\mathbf{A}^\sigma(\mathbf{k}, \omega, \mathbf{r}, t)$ are found such that H^σ, $\mathbf{A}^\sigma$ are independent and $\{H^\sigma, A_i^\sigma\}_8 = \{A_i^\sigma, A_j^\sigma\}_8 = 0$ for $i,\ j = 1,\ 2,\ 3$, $\{\cdot,\cdot\}_8$ denoting the Poisson brackets in the eight-dimensional phase space [6, 7]. The procedure to find a complete solution is the following: since H^σ, $\mathbf{A}^\sigma$ are four independent functions, the system $\alpha = H^\sigma$, $\mathbf{a} = \mathbf{A}^\sigma$ can be inverted to yield $\mathbf{k} = \bar{\mathbf{k}}^\sigma(\mathbf{a}, \alpha, \mathbf{r}, t)$, $\omega = \bar{\omega}^\sigma(\mathbf{a}, \alpha, \mathbf{r}, t)$; therefrom the differential form $\bar{\mathbf{k}}^\sigma \cdot d\mathbf{r} - \bar{\omega}^\sigma dt$ is found to be exact so that $\psi^\sigma = \int_\gamma(\bar{\mathbf{k}}^\sigma \cdot d\mathbf{r} - \bar{\omega}^\sigma dt)$ for an arbitrary path γ in the $(\mathbf{r}, t)$ configuration space. Note that the condition **(ii)** is automatically fulfilled. Hence the solution of Eq.(7) relevant to the mode σ is,

$$\underline{\underline{W}}^{(0)\sigma} = 2\pi\delta(\alpha)Q^\sigma(\mathbf{a}, \mathbf{r}, t)\hat{\mathbf{e}}^\sigma\hat{\mathbf{e}}^{\sigma *}, \tag{10}$$

where Q^σ is to be determined, the corresponding eikonal ψ^σ being computed from the chosen Hamiltonian H^σ. In the following we will consider specifically $H^\sigma = \omega - \Omega^\sigma(\mathbf{k}, \mathbf{r}, t)$, for which $\psi^\sigma = S^\sigma(\mathbf{a}, \mathbf{r}, t) - \alpha t$, with $S^\sigma(\mathbf{a}, \mathbf{r}, t)$ solution of the Hamilton-Jacobi equation in the six-dimensional $(\mathbf{k}, \mathbf{r})$ phase space, $\frac{\partial}{\partial t}S^\sigma + \Omega^\sigma(\frac{\partial}{\partial \mathbf{r}}S^\sigma, \mathbf{r}, t) = 0$, Ω^σ being the corresponding Hamiltonian, and $\mathbf{k} = \frac{\partial S^\sigma(\mathbf{a},\mathbf{r},t)}{\partial \mathbf{r}}$. Turning now to Eqs.(8) and (9), Eq.(8) establishes that the work done by $\mathbf{j}$ on $\mathbf{E}^{(0)}$ is simply zero. As for Eq.(9), it yields

$$\frac{\partial}{\partial t}W^\sigma(\mathbf{a}, \mathbf{r}, t) + \nabla\cdot\left(\mathbf{v}_g^\sigma(\mathbf{a}, \mathbf{r}, t)W^\sigma(\mathbf{a}, \mathbf{r}, t)\right) = -2\gamma^\sigma(\mathbf{a}, \mathbf{r}, t)W^\sigma(\mathbf{a}, \mathbf{r}, t) + P^\sigma(\mathbf{a}, \mathbf{r}, t), \tag{11}$$

which describes the transport of the wave energy density, $W^\sigma(\mathbf{a}, \mathbf{r}, t) \equiv \left[\frac{1}{\omega}\frac{\partial\omega^2\varepsilon^\sigma}{\partial\omega}\right]\frac{Q^\sigma}{4\pi V\bar{T}}$ (averaged aver phases, volume V and time $\bar{T}$), along the streamlines of the group velocity $\mathbf{v}_g^\sigma(\mathbf{a},\mathbf{r},t) = \left[\frac{\partial\Omega^\sigma}{\partial\mathbf{k}}\right]_{\mathbf{k}=\bar{\mathbf{k}}^\sigma}$, accounting for both absorption (γ^σ) and emission (P^σ), $(\gamma^\sigma, P^\sigma) = \left[\frac{1}{\omega}\frac{\partial\omega^2\varepsilon^\sigma}{\partial\omega}\right]^{-1}\left(\omega\varepsilon_{a,\ ij}\ \hat{e}_i^{\sigma *}\hat{e}_j^\sigma,\ 4\pi\frac{|\hat{\mathbf{e}}^{\sigma *}\cdot\mathbf{j}|^2}{VT}\right)$, with $\varepsilon^\sigma = \varepsilon_{h,\ ij}\ \hat{e}_i^{\sigma *}\hat{e}_j^\sigma$ the effective dielectric function, [8]. For $P^\sigma = 0$, the conventional GO single quasi-plane wave solution corresponds to $W^\sigma(\mathbf{a},\mathbf{r},t) = \delta(\mathbf{a}-\mathbf{a}_0)w^\sigma(\mathbf{r},t)$, where $\mathbf{a}_0$ labels the particular quasi-plane wave and $w^\sigma(\mathbf{r},t)$ is the corresponding wave energy density in the ordinary $\mathbf{r}$-space. Since $(\mathbf{a},\mathbf{r})$ are not canonically conjugate coordinates, Eq.(11) is not, in general, in the form of a kinetic equation. In terms of the conjugate momenta $\mathbf{k} = \frac{\partial S^\sigma}{\partial\mathbf{r}}(\mathbf{a},\mathbf{r},t)$ the transport equation (11) becomes

$$\frac{\partial}{\partial t}W_{ph}^\sigma + \{\Omega^\sigma, W_{ph}^\sigma\} = \left(\Gamma^\sigma - 2\gamma^\sigma\right)W_{ph}^\sigma + P_{ph}^\sigma, \tag{12}$$

where the label "*ph*" distinguishes the distribution in the $(\mathbf{k},\mathbf{r})$ canonical coordinates from the corresponding distribution in the $(\mathbf{a},\mathbf{r})$ non-canonical coordinates. In (12) $\Gamma^\sigma(\mathbf{k},\mathbf{r},t)$ accounts for the variations, with respect to both $\mathbf{r}$ and t, of the Jacobian of the transformation $\mathbf{a}\mapsto\mathbf{k}$. If $\Gamma^\sigma = \frac{1}{\Omega^\sigma}\frac{\partial\Omega^\sigma}{\partial t}$, Eq.(12) agrees with the results obtained in Ref.[5].

In conclusion, the generalized Fourier integral representation of the wave field [3,4] has been employed to address the problem of the energy transport for wave packets propagating in weakly inhomogeneous media, including local sources and dissipation. The wave kinetic equation governing the wave energy density in $(\mathbf{k},\mathbf{r})$-phase space has been obtained with an ordering procedure consistent with both the conventional geometrical optics and the method based on the Weyl representation for electromagnetic waves [5].

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