

## Analytic Approximated Solution for the Bohm Sheath Potential

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### ABSTRACT

The Bohm treatment of plasma potential leads to a nonlinear first order differential equation which can only be solved numerically [1,2,3]. In this paper we have determined the coefficients of the powers  $\phi^2$ ,  $\phi^3$  and  $\phi^4$ . The square root of this expansion gives the derivative of the potential. This differential equation can be solved analytically, obtaining first the position as a function of the potential. The important point of this result is that in this case the inverse transformation can be also performed analytically, giving the potential as a function of position. This result is compared with the numerical solution of the differential equation and the agreement is very good for most of the values of the characteristic parameters. Approximated potentials using third degree expansions have been also determined and the accuracy of this approximation is acceptable, but not as good as the fourth degree approach, previously described. In all the range of the parameters, the potentials here found have better accuracy than previous approximations [2].

### Section I. INTRODUCTION

The classic Bohm's treatment [1] of plasma potential near a wall allows to obtain the position as a function of the potential, however the most usual need is to determine the potential as a function to the distance to the wall [2,3]. This inversion function can only be determined by numerical calculations. The idea of the present paper is to find an analytic formula for the potential as a function of the distance using the recently developed method of quasi fractional approximations [4,5]. A fourth order approximation can be obtained with this method combining exponential with rational functions, which is very precise and can be used for most of the experimental works with enough accuracy. The theoretical model is presented in Section II, the results are discussed in Section III and finally, section IV is devoted to the conclusions.

### Section II. THEORETICAL MODEL

The Bohm model for the plasma potential near the wall is given by the Poisson equation

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{ne}{\epsilon_0} \left( \exp\left(\frac{e\varphi}{T}\right) - \sqrt{\frac{\frac{1}{2}mv^2}{\frac{1}{2}mv^2 - e\varphi}} \right) \quad (1)$$

Using dimensionless variables

$$y = \frac{x}{\lambda_D}, \quad K = \frac{\frac{1}{2}mv^2}{T}, \quad \phi = -\frac{e\varphi}{T} \quad (2)$$

the equation (1) can be integrated giving

$$\psi = \sqrt{\int \left( e^\phi - \sqrt{\frac{K}{K+\phi}} \right) d\phi} \quad (3)$$

with

$$\psi = \frac{\partial \phi}{\partial y} \quad (4)$$

The sheath potential can be calculated by a second integration

$$\int_{\phi_w}^{\phi} \frac{d\phi}{\sqrt{F}} = \sqrt{\frac{2K-1}{4K}} y \quad (5)$$

with

$$F = 2K \sqrt{1 + \frac{\phi}{K}} + e^{-\phi} - 2K - 1 \quad (6)$$

Here for  $y = 0$ , the potential takes its value at the wall  $\phi_w$ . The power series for  $F$  is

$$F \simeq \left( \frac{1}{2} - \frac{1}{4K} \right) \phi^2 + \left( \frac{1}{8K} - \frac{1}{6} \right) \phi^3 + \left( \frac{1}{24} - \frac{5}{64K^3} \right) \phi^4 + \dots \quad (7)$$

The usual solution is obtained by keeping only the first term of the series, giving

$$\phi_2 = \phi_w \exp \left( -\frac{1}{2} \sqrt{\frac{2K-1}{K}} y \right) \quad (8)$$

Taking  $F$  in the form :

$$\hat{F} \simeq \phi^2 (1 + \alpha\phi + \gamma\phi^2) \quad (9)$$

where  $\alpha$  is

$$\alpha = \frac{3 - 4K^2}{12K^2 - 6K} \quad (10)$$

and  $\gamma$  is determined by the potential at the wall :

$$\hat{F}(\phi_w) = F(\phi_w) \quad (11)$$

$$\gamma = -\frac{4K \left( 1 - e^{-\phi_w} + 2K - 2K \sqrt{1 + \frac{\phi_w}{K}} + \frac{\phi_w^2}{4K} (2K - 1) (1 + \alpha\phi_w) \right)}{(2K - 1)\phi_w^4} \quad (12)$$

Now the integral equation is obtained as

$$\int -\frac{d\phi_4}{\phi_4 \sqrt{1 + \alpha\phi_4 + \gamma\phi_4^2}} = \sqrt{\frac{2K-1}{4K}} y \quad (13)$$

This integral can be solved, giving

$$\frac{2 + \alpha\phi_4 - 2\sqrt{1 + \alpha\phi_4 + \gamma\phi_4^2}}{\phi_4} = Ce^{-\frac{1}{2}\sqrt{\frac{2K-1}{K}}y} \quad (14)$$

$$\phi_4 = \frac{4Ce^{-\frac{1}{2}\sqrt{\frac{2K-1}{K}}y}}{\left[Ce^{-\frac{1}{2}\sqrt{\frac{2K-1}{K}}y} - \alpha\right]^2 - 4\gamma} \quad (15)$$

The constant of integration  $C$  is determined by the boundary condition at the wall

$$C = \frac{2 + \alpha\phi_w - 2\sqrt{1 + \alpha\phi_w + \gamma\phi_w^2}}{\phi_w} \quad (16)$$

This fourth degree approximation can be rearranged in order to be well behaved for any value of  $K$  and  $\phi_w$ , giving

$$\phi_4 = \frac{24e^{-\frac{1}{2}\sqrt{\frac{2K-1}{K}}y}K[(3 - 4K^2)\phi_w + C_4\sqrt{2K-1}\phi_w^3]}{\phi_w^2 \left[ \frac{(3-4K^2)\phi_w}{\sqrt{2K-1}} \left( e^{-\frac{1}{2}\sqrt{\frac{2K-1}{K}}y} - 1 \right) + C_4e^{-\frac{1}{2}\sqrt{\frac{2K-1}{K}}y} \right]^2 + 576K^3C_5} \quad (17)$$

By this way  $\phi_4$  is also finite at  $K = 1/2$ , and the coefficients  $C_4$  and  $C_5$  are functions of the wall potential

$$C_4 = 12K\sqrt{2K-1} - 12K\sqrt{2K-1 + \frac{(3-4K^2)\phi_w}{6K} - \frac{4KC_5}{\phi_w^2}} \quad (18)$$

$$C_5 = 1 - e^{-\phi_w} + 2K - 2K\sqrt{\frac{1+\phi_w}{K}} + \frac{\phi_w^2}{4K} \left( 2K - 1 + \frac{(3-4K^2)\phi_w}{6K} \right) \quad (19)$$

A similar procedure can be used to obtain an approximant using only third degree powers in  $\phi$ , but the accuracy is low in this case, as we show in the next section. The details to obtain  $\phi_3$  will be given elsewhere.

### Section III. RESULTS

In figures 1 and 2 we show the potential  $\phi$  obtained by: 1) numerical calculation (dotted line); 2) using our fourth order approximation  $\phi_4$ ; 3) the third order approximant  $\phi_3$ , and the usual second order approximation which appears in previous publications. Figure 1 shows the case  $K = 1/2$ , and  $K = 1$  is shown in figure 2. The accuracy of the approximations is higher in case  $K = 1$  than for  $K = 1/2$ . The accuracy of the approximant here presented is always higher than previous ones. In order to get a global idea of the accuracy of the approximants we show in figure 3 the maximum percent error of each approximation as a function of the wall potential. Figure 4 shows the absolute value of the maximum percent error as a function of the parameter  $K$ , for  $\phi_w = 4$ . To be specific, we run curves similar to those shown in figures 1 and 2 for determined values of  $K$ , and we

select the highest percent error in each figure, in order to build the figure 4. The same procedure is followed to produce figure 3.

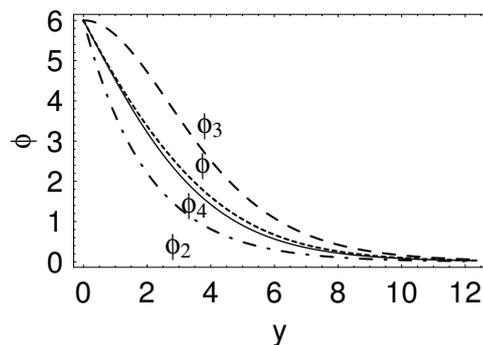
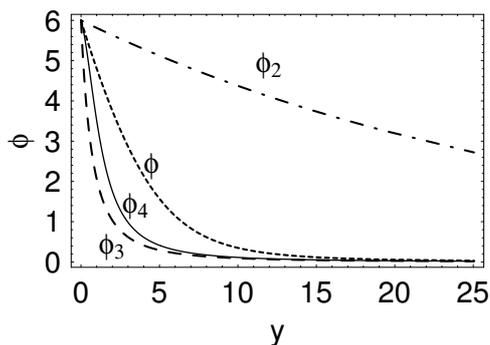


Fig. 1 Potential versus distance ( $K = 1/2$ ). Fig. 2 Potential versus distance ( $K = 1$ ).

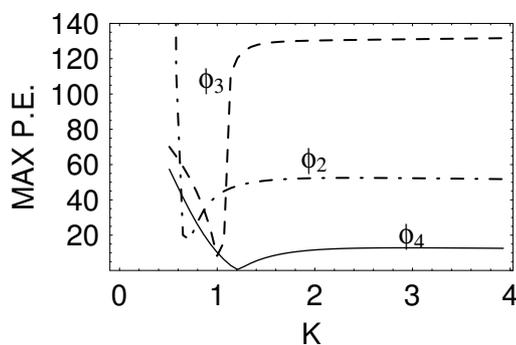
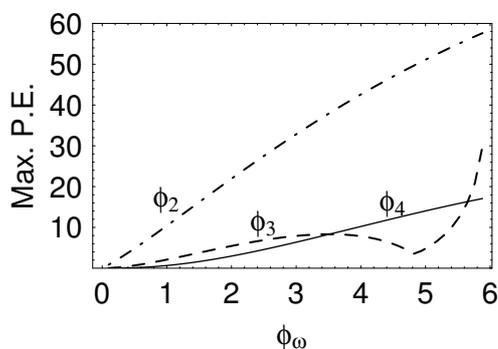


Fig. 3 Maximum percent error vs.  $\phi_w$  ( $K=1$ ). Fig. 4 Maximum percent error vs.  $K$ .

#### Section IV. CONCLUSION

We have obtained an explicit approximated algebraic expression for the plasma potential near a wall, which allows the direct calculation of the sheath potential as a function of the distance to the wall. This potential is given as a combination of exponential functions with rational functions. The accuracy of this approximated potential is high and will be enough for most of the experimental works. The highest error is always at the intermediate region, and the approximated potential always matches very well for values near the wall and far away.

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