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Turbulent Diffusion in External Magnetic Field: Time-Nonlocal Effects and Numerical Simulations

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I. Introduction

In Ref.[1] we formulated the basic points of the statistical theory of turbulent transport giving the time-nonlocal generalization of the Dupree-Weinstock renormalized theory. Then the similar generalization was proposed for the description of particle diffusion in magnetized plasmas under drift wave turbulence [2]. Kinetic equations for distribution functions and renormalized transition probability were derived and general relations for time-nonlocal transport were obtained.

The purpose of the present contribution is to apply the theory developed in Ref. [2] to more specific studies of plasma transport in the case of drift wave turbulence and to compare the analytical estimates with the numerical simulations. The main attention is paid to the model of quasi two-dimensional turbulence.

II. Diffusion across external magnetic field (analytical estimates)

It follows from the general relations describing turbulent diffusion in external magnetic field within the drift-kinetic approximation [2] that in the case of two-dimensional model the renormalized transition probability $\left\langle W(\mathbf{r},\mathbf{r}';t-t')\right\rangle$ calculated with regard to turbulent field influence on particle motion satisfies the following equation

$$\frac{\partial}{\partial t}\left\langle W(\mathbf{r},\mathbf{r}';t,t')\right\rangle = \int_{t'}^{t} dt'' D_{ij}(t,t'') \frac{\partial^2 W(\mathbf{r},\mathbf{r}';t'',t')}{\partial r_i \partial r_j} \tag{1}$$

with the initial condition

$$W(\mathbf{r},\mathbf{r}';t',t') = \delta(\mathbf{r}-\mathbf{r}'). \tag{2}$$

Here $D_{ij}(t,t')$ is the time-nonlocal diffusion coefficient given by

$$D_{ij}(t,t') = \left(\frac{e}{m}\right)^2 \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{k^2}{\Omega^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \left\langle \delta\Phi(t)\delta\Phi(t')\right\rangle_{\mathbf{k}} \left\langle W_{\mathbf{k}}(t',t)\right\rangle, \tag{3}$$

Ω is the cyclotron frequency. In these relations the vector quantities are assumed to be two-dimensional and thus $i, j = x, y$.

In the case of stationary turbulent spectrum the formal solution of Eq.(1) is the following

$$W(\mathbf{r},\mathbf{r}';t,t') = \int \frac{d\mathbf{k}}{(2\pi)^2} \int \frac{d\omega}{2\pi} e^{i\mathbf{k}\mathbf{R} - i\omega\tau} W_{\mathbf{k}\omega}, \tag{4}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $\tau = t - t'$,

$$W_{\mathbf{k}\omega} = \frac{i}{\omega + i k_i k_j D_{ij\,\omega}}, \tag{5}$$

and thus, the Fourier-component of the diffusion coefficient satisfies the equation

$$D_{ij\,\omega} = \left(\frac{e}{m}\right)^2 \int \frac{d\mathbf{k}}{(2\pi)^2} \int \frac{d\omega'}{2\pi} \frac{k^2}{\Omega^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{i\left\langle \delta\Phi^2 \right\rangle_{\mathbf{k}\omega'}}{\omega - \omega' + ik_i k_j D_{ij\,\omega-\omega'}}\,. \tag{6}$$

Using the solution (4) it is easy to calculate meansquare displacements

$$\left\langle \Delta r_i \Delta r_j \right\rangle_\tau = -\left(1+\delta_{ij}\right) \int \frac{d\omega}{2\pi} \frac{D_{ij\,\omega} e^{-i\omega\tau}}{(\omega + i0)^2}\,. \tag{7}$$

As is seen, meansquare displacements are determined by the pole $\omega = 0$ responsible for the conventional diffusion (meansquare displacement is proportional to τ) and by the singularities of $D_{ij\,\omega}$ which are determined by the turbulent field spectrum. In the case of spectrum generating $D_{ij\,\omega} \sim (-i\omega)^{1-\alpha}$, $(0 < \alpha < 1)$ equation (7) describes "strange" diffusion with meansquare displacement proportional to τ^α.

Postulating the correlation function of the turbulent field potential to be

$$\left\langle \delta\Phi(\mathbf{r},t)\delta\Phi(\mathbf{r}',t') \right\rangle = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{-\frac{k_x^2}{\Delta k_x^2}} e^{-\frac{(k_y^2 - k_{y0}^2)^2}{\Delta k_y^2}} \frac{4\pi}{\Delta k_x \Delta k_y} \delta\Phi_0^2 \cos(\mathbf{kR} - \omega_{\mathbf{k}}\tau), \tag{8}$$

where $\omega_{\mathbf{k}}$ is the eigenfrequency of the mode, one obtains

$$D_{ij\,\omega} = \left(\frac{e}{m}\right)^2 \frac{\delta\Phi_0^2}{\Omega^2} \frac{2\pi}{\Delta k_x \Delta k_y} \sum_s \int \frac{d\mathbf{k}}{(2\pi)^2} k^2 \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{-\frac{k_x^2}{\Delta k_x^2} - \frac{(k_y^2 - k_{y0}^2)^2}{\Delta k_y^2}} \frac{i}{\omega - s\omega_{\mathbf{k}} + ik_i k_j D_{ij\,\omega - s\omega_{\mathbf{k}}}}\,. \tag{9}$$

In the general case this equation should be solved numerically. However, under the assumption that the ω-dependence of the diffusion coefficient is weak, it is possible to find approximate analytical solutions. In such case Eqs. (7), (9) predict the time-dependence of the meansquare displacement of the type

$$\left\langle \Delta r_i \Delta r_j \right\rangle_\tau = \int \frac{d\mathbf{k}}{(2\pi)^2} \left[A_{ij}\tau + B_{ij}\left(1 - e^{\Delta_1 \tau}\cos\omega_{\mathbf{k}}\tau\right) + C_{ij}\left(1 - e^{\Delta_2 \tau}\cos\omega_{\mathbf{k}}\tau\right) \right], \tag{10}$$

where $\Delta_1 \simeq -2\left(1 + i\Gamma_0'\right)/\Gamma_0''$, $\Delta_2 = \Gamma_0/\left(1 + i\Gamma_0'\right)$, Γ_0, Γ_0' and Γ_0'' are the zero-frequency values of $\Gamma_\omega = k_i k_j D_{ij\,\omega}$ and its frequency derivatives, respectively. These quantities, as well as coefficients A_{ij}, B_{ij} and C_{ij}, should be determined using ω-expansion of Eq. (9).

In the case of narrow spectrum ($\Delta k_x, \Delta k_y < k_{y0}$) Eq. (9) generates the following relation

$$\Gamma_\omega = \frac{1}{2}\tilde{\Omega}^2 \sum_{s=\pm 1} \frac{i}{\omega - s\omega_{\mathbf{k}} + i\Gamma_{\omega - s\omega_{\mathbf{k}_0}}} \tag{11}$$

where

$$\tilde{\Omega}^2 = \left(\frac{e}{m}\right)^2 \frac{\delta\Phi_0^2}{\Omega^2} \Delta k_x^2 k_{y0}^2\,.$$

Analysis of this relation shows that in the case of weak time-nonlocality the diffusion is generated above the threshold value of potential $\delta\Phi_{th}$ which is influenced by the time-nonlocality.

III. Numerical simulations

We study two-dimensional particle diffusion in the external magnetic field ($\mathbf{B}_0 = \mathbf{e}_z B_0$) and the prescribed electric field taken as the superposition of $2N^2$ modes

$$\delta\Phi(\mathbf{r},t) = \sum_{i=1}^{N}\sum_{j=1}^{2N} \delta\Phi_{ij} \cos\left(\omega_{\mathbf{k}\,ij} t - k_{xi}x - k_{yj}y + \alpha_i + \beta_j\right), \tag{12}$$

where α_i and β_j are the random phases,

$$\omega_{\mathbf{k}\,ij} = \frac{L_n^{-1} k_{yj} c_s^2}{(1 + k_{xi}^2 \rho_s^2 + k_{yj}^2 \rho_s^2)} \tag{13}$$

is the drift wave frequency; c_s - ion sound velocity, $\rho_s = c_s/\Omega$, $L_n = -[d \ln n/dx]^{-1}$ - inhomogeneity scale length of density. The spectrum of the potential is taken in the form

$$\delta\Phi_{ij}^2 = \delta\Phi_0^2 \frac{25}{\pi N^2} \exp\left[-\left(\frac{k_{xi}}{\Delta k_x}\right)^2 - \left(\frac{k_{yj}}{\Delta k_y} - 2.5\right)^2\right], \tag{14}$$

$$k_{xi} = 2.5 i \Delta k/N, \qquad i = 1,\ldots,N;$$

$$k_{yj} = 2.5 j \Delta k/N, \qquad j = 1,\ldots,2N. \tag{15}$$

Particle $E \times B$ drift trajectories were calculated numerically for different realizations of random phases and then the meansquare displacements $\langle \Delta x^2 \rangle$ and $\langle \Delta y^2 \rangle$ were found as average over realizations. The results of simulation are given in Figs. 1-6. The dimensionless inhomogeneity scale length was taken $L_n/\rho_s = 200$. The time is normalized by $2\pi L_n/c_s$ and distance by $2\pi\rho_s$. For dimensionless potential and spectrum width we use the notation $\sigma = \left(e/mc_s^2\right)\delta\Phi_0$, $\kappa = \Delta k \rho_s$.

IV. Conclusions

Both analytical estimates and numerical simulations show the following peculiarities of the two-dimensional turbulent diffusion across external magnetic field:

1. Diffusion starts from the threshold value of the turbulent field amplitude $\delta\Phi_{th}$ which is inversely proportional to the spectral width and is different for x- and y-direction. In the general case this value should be estimated with regard to time-nonlocal effects Eq.(11).
2. Different fashion of diffusion in two directions is caused by difference in dependencies of eigenfrequency and field spectrum on k_x and k_y.
3. For $\delta\Phi_0 < \delta\Phi_{th}$ the behavior of $\langle \Delta x^2 \rangle_\tau$ is characterized by small oscillation with no considerable deviation from the starting point (Figs. 4-6). Diffusion in the y-direction is nevertheless observed for the same values of spectral width κ and potential σ. On the initial stage (which duration depends on κ and σ) it characterized by parabolic time-dependence of $\langle \Delta y^2 \rangle_\tau$. The latter could be associated with time-nonlocal effects.

4. Particle diffusion in both directions is accompanied by frequent small oscillations, not always seen in large scale, which also are the manifestation of the time-nonlocality.

References

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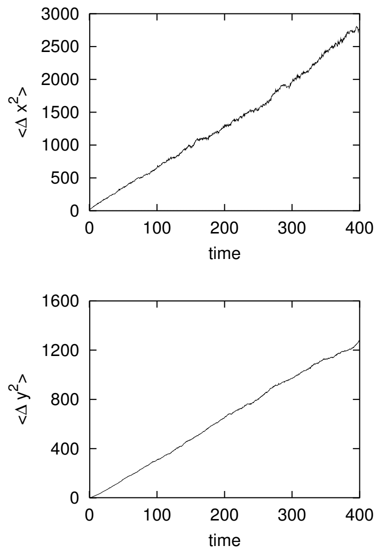

Fig. 1. $\kappa = 0.4$, $\sigma = 0.1$

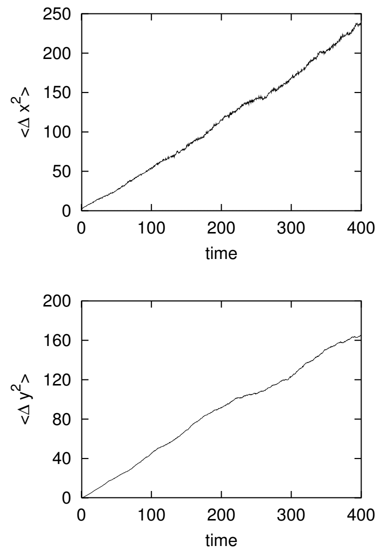

Fig. 2. $\kappa = 0.4$, $\sigma = 0.01$

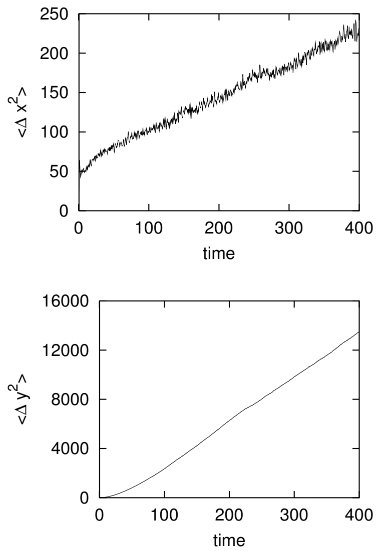

Fig. 3. $\kappa = 0.1$, $\sigma = 0.1$

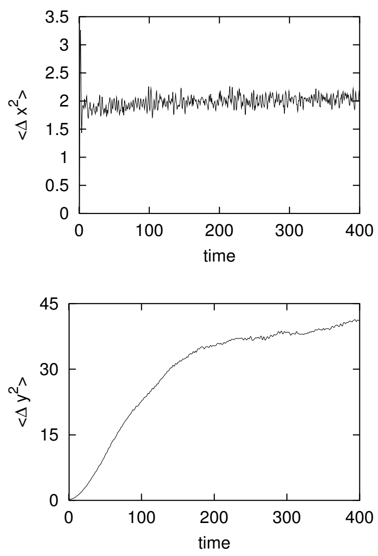

Fig. 4. $\kappa = 0.1$, $\sigma = 0.01$

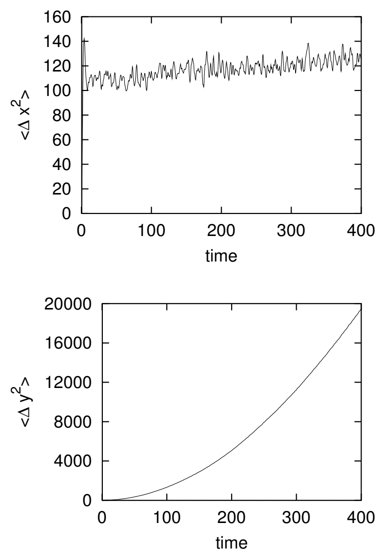

Fig. 5. $\kappa = 0.04$, $\sigma = 0.1$

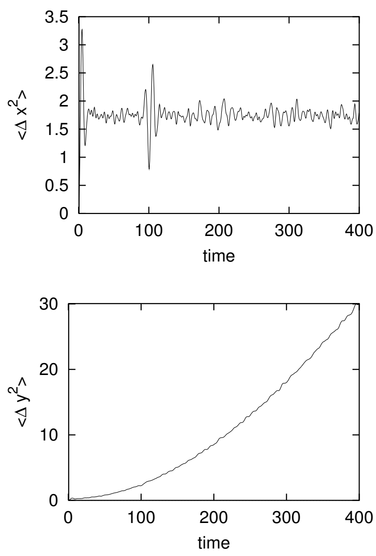

Fig. 6. $\kappa = 0.04$, $\sigma = 0.01$