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Localization of fast magnetosonic eigenmodes in spherical tori

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Edge-localized fast magnetosonic eigenmodes (FME) may be responsible for the observed sub-ion cyclotron emission (SICE) in recent NSTX experiments, [1,2]. These modes can be driven unstable by resonant interaction with a small population of energetic ions, having an anisotropic distribution in velocity space. Radially localized modes are important not only to explain the observed SICE, but also because these modes might open a possibility for transferring energy from the fusion products to the background ions.

The observation of sub-ion cyclotron emission in NSTX at frequencies about half of the ion cyclotron frequency, calls for an extension of the previous eigenmode analysis to be valid also this frequency range. In the present paper, we extend the eigenmode analysis to be valid for spherical tokamak geometry and sub-ion cyclotron frequencies. The radial and poloidal structure of these eigenmodes is analyzed, by solving the eigenmode equation using a variational approach.

The eigenmode equation

The eigenmode equation for the perturbed magnetic field in a cold, inhomogeneous and magnetized plasma with one ion species can be obtained from the Maxwell equations by assuming the perturbed quantities X to depend on time as $\exp(-i\omega t)$, ω is the wave frequency, and by introducing the elliptic-toroidal co-ordinates ρ, ϑ and φ, where ρ is a radial co-ordinate such that $\rho = \sqrt{(\sqrt{x^2+y^2}-R_0)^2+(z/\kappa)^2}$, $\vartheta = \arctan[z/\kappa, \sqrt{x^2+y^2}-R_0]$ is the modified poloidal angle and $\varphi = \arctan[x,y]$ is the toroidal angle, κ is the ellipticity of the flux surface defined as the ratio of major and minor radius of the ellipse, R_0 is the major radius, $\arctan[x,y] = \arctan x/y + \pi(1-H(y))$ and $H(y)$ is the Heaviside function. Assuming $E_\varphi \simeq 0$ and toroidal variation as $X \propto \exp(-in\varphi)$, where n is the toroidal mode number, we can use Ampére's law to express E_ρ and E_ϑ in terms of B_φ. Then, taking the third component of Faraday's law, we arrive at the following eigenmode equation for B_φ

$$\frac{\partial}{\partial\rho}\left\{\frac{1}{c_n}\left(a_3\frac{\partial B_\varphi}{\partial\rho}+a_1\frac{\partial B_\varphi}{\partial\vartheta}\right)\right\}+\frac{\partial}{\partial\vartheta}\left\{\frac{1}{c_n}\left(a_4\frac{\partial B_\varphi}{\partial\rho}+a_2\frac{\partial B_\varphi}{\partial\vartheta}\right)\right\}=\frac{i\omega}{c}\frac{\sqrt{g}}{R^2}B_\varphi, \qquad (1)$$

where the constants a_j are

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \epsilon_{11}\frac{\omega}{c}\begin{pmatrix} -R\omega/\omega_{ci} - i\sqrt{g}g^{12}K_n & -ig^{22}\sqrt{g}K_n \\ -ig^{11}\sqrt{g}K_n & R\omega/\omega_{ci} - i\sqrt{g}g^{21}K_n \end{pmatrix} \qquad (2)$$

$c_n = a_1a_4 - a_2a_3$, $K_n = 1 + n^2v_A^2(\omega^2 - \omega_{ci}^2)/(R^2\omega_{ci}^2\omega^2)$ and $g = \det g_{ij} = \rho^2R^2\kappa^2$, g_{ij} is the metric tensor, given in [3], and $R = R_0 + \rho\cos\vartheta$. The dielectric tensor elements can be simplified to $\epsilon_{11} = \epsilon_{22} = \omega_{pi}^2/(\omega_{ci}^2 - \omega^2)$, $\epsilon_{12} = -\epsilon_{21} = i\epsilon_{11}\omega/\omega_{ci}$ where ω_{pi} is the ion plasma frequency. We introduce $B_\varphi(\rho,\vartheta) = \hat{B}(\rho,\vartheta)e^{imf(\vartheta)}$, and rewrite Eq. (1) to

$$\frac{i\omega R^2c}{\sqrt{g}v_A^2}\left\{\frac{\partial}{\partial\rho}\left[\frac{1}{c_n}\left(a_3\frac{\partial\hat{B}}{\partial\rho} + a_1\frac{\partial\hat{B}}{\partial\vartheta}\right)\right] + \frac{\partial}{\partial\vartheta}\left[\frac{1}{c_n}\left(a_4\frac{\partial\hat{B}}{\partial\rho} + a_2\frac{\partial\hat{B}}{\partial\vartheta}\right)\right] + \frac{imf'(\vartheta)}{c_n}\left[(a_1 + a_4)\frac{\partial\hat{B}}{\partial\rho} + 2a_2\frac{\partial\hat{B}}{\partial\vartheta}\right]\right\} + V(\rho,\vartheta)\hat{B} = 0, \quad (3)$$

with the potential $V(\rho,\vartheta) = iV_{im} + V_{re}$, where

$$V_{re}(\rho,\vartheta) = \frac{\omega^2}{v_A^2} - \frac{m^2}{\kappa^2\rho^2}\frac{(\omega^2 - \omega_{ci}^2)K_n}{\omega^2 - \omega_{ci}^2K_n^2} - \frac{m\omega R^2f'(\vartheta)}{v_A^2\sqrt{g}}\frac{\partial}{\partial\rho}\left(\frac{v_A^2(\omega^2 - \omega_{ci}^2)}{R\omega_{ci}(\omega_{ci}^2K_n^2 - \omega^2)}\right) \quad (4)$$

and

$$iV_{im} = -\frac{m\omega R^2c}{\sqrt{g}v_A^2}\left\{f''(\vartheta)\frac{a_2}{c_n} + if'(\vartheta)\mathrm{Im}\left[\frac{\partial}{\partial\rho}\left(\frac{a_1}{c_n}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{a_2}{c_n}\right)\right]\right\}, \quad (5)$$

We assume that κ is constant as a function of ρ and $f(\vartheta)$ is chosen so that $f'(\vartheta)^{-2} = \cos^2\vartheta + \kappa^2\sin^2\vartheta$, the prime denotes a derivative with respect to ϑ.

In the above equation, toroidal effects have been included in v_A and ω_{ci} through the poloidal dependence of the equilibrium magnetic field. The last term in Eq. (4) originates from the Hall term and breaks the poloidal symmetry even when ϵ is negligibly small [4]. This term is large at the edge, where the plasma density profile is steep and its derivative is very large. In deriving Eq. (3), we have neglected the mode coupling associated with the ellipticity. This approximation is justified in the limit when the ellipticity is not too large, i.e. $(\kappa^2 - 1)/(\kappa^2 + 1)/2 \ll 1$, which is well satisfied in most of cases of interest.

Variational analysis

For real $\hat{B}$, the Lagrangian corresponding to Eq. (3) is given by

$$L = \xi\left[g^{11}\left(\frac{\partial\hat{B}}{\partial r}\right)^2 + g^{22}\left(\frac{\partial\hat{B}}{\partial\vartheta}\right)^2 + 2g^{12}\left(\frac{\partial\hat{B}}{\partial r}\right)\left(\frac{\partial\hat{B}}{\partial\vartheta}\right)\right] + H(r,\vartheta)\hat{B}^2$$

where $\xi = -\sqrt{g}K_n\epsilon_{11}/c_n$, $H = V_{re}/A$ and $A = (\omega R)^2/(\sqrt{g}v_A^2)$. Assuming an ansatz function of the form $\hat{B}(\rho,\vartheta) = B_0\exp(-(\rho-\rho_0)^2/(2\Delta^2))\exp(-(\vartheta-\vartheta_0)^2/(2\eta^2))$, the variational principle $\delta\int d\rho d\vartheta L(\rho_0,\Delta,\vartheta_0,\eta) = 0$ determines the localization radius ρ_0, the radial localization width Δ, the localization angle ϑ_0 and the poloidal localization width η. The variations with respect to ρ_0 and ϑ_0 imply that

$$\left.\frac{\partial H}{\partial\rho}\right|_{\rho_0,\vartheta_0} = 0 \quad \text{and} \quad \left.\frac{\partial H}{\partial\vartheta}\right|_{\rho_0,\vartheta_0} = 0 \quad (6)$$

and to find the rest of the parameters we expand $H(\rho,\vartheta,\omega)$ around ρ_0 and ϑ_0. Then, the variations with respect to B_0^2, Δ and η give the following set of equations

$$\Delta^4 = \frac{2\xi\left[(1+\kappa^{-2})/2 + e^{-\eta^2}(1-\kappa^{-2})/2\right]}{\partial^2 H/\partial\rho^2|_{\rho_0,\vartheta_0}} \tag{7}$$

$$\eta^4 = \frac{2\xi\left[(1+\kappa^{-2})/2 + e^{-\eta^2}(\eta^2+1)(\kappa^{-2}-1)/2\right]}{\rho_0^2[\partial^2 H/\partial\vartheta^2|_{\rho_0,\vartheta_0} - 2\xi(1-\kappa^{-2})e^{-\eta^2}(1/(2\Delta^2)+1/\rho_0^2)]} \tag{8}$$

$$H(\rho_0,\vartheta_0,\omega) + \frac{\xi(1+\kappa^{-2})}{2\Delta^2}\left(1+\frac{\Delta^2}{\rho_0^2\eta^2}\right) = \frac{e^{-\eta^2}\xi(\kappa^{-2}-1)}{2}\left(\frac{1+\eta^2}{\Delta^2} + \frac{\eta^2-2+2\eta^4}{2\rho_0^2\eta^2}\right) \tag{9}$$

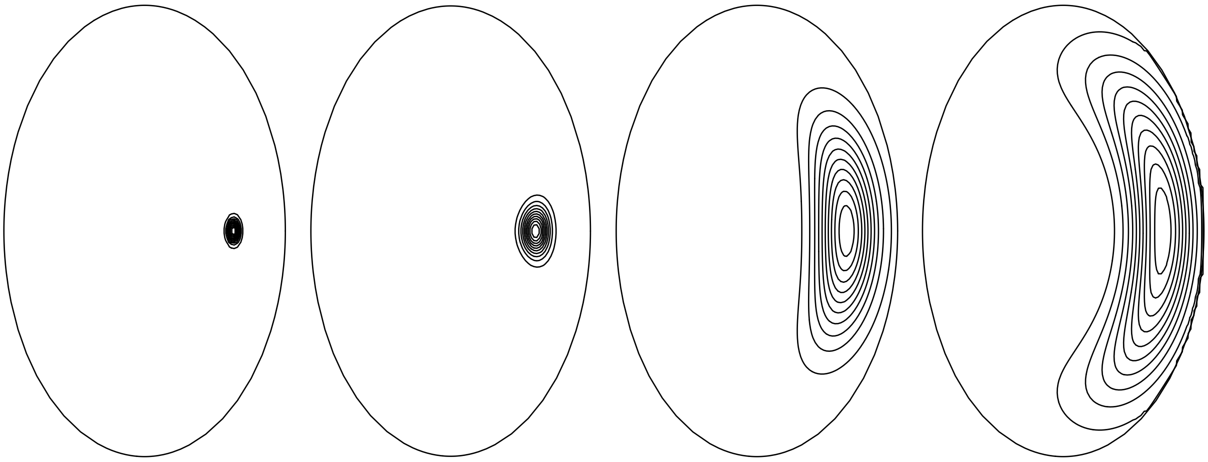
Equations (6-9) determine the eigenmode frequency ω_0, localization radius ρ_0 and angle ϑ_0, and poloidal and radial localization widths (η and Δ).

Numerical results

We have calculated the localization angle and radius and eigenmode frequency for the following parameters relevant to the NSTX experiments: ellipticity $\kappa = 1.6$, major radius $R_0 = 85$ cm, density profile $n(\rho) = (1-(\rho/a)^2)^\sigma$, with $\sigma = 0.5$ or $\sigma = 1$, minor radius $a = 65$ cm, ion cyclotron frequency at the edge $\omega_{ci} = 14.5$ MHz and Alfvén velocity at the edge $v_{A,edge} = 10^8$. The magnetic field is approximately $B = B_0R_0/R$ in the low to medium beta NSTX plasmas [2], where B_0 is the magnetic field at the plasma center. The experimentally observed peaks appear in two bands, spanning $0.7-1.2$ MHz and $1.5-2.2$ MHz. The peaks are separated by a spacing of about 120 kHz, and some of the peaks consist of subpeaks separated by about 20 kHz.

We have solved numerically the system of equations (6)-(9) and found localized solutions with frequencies $0 < \omega < \omega_{ci}$, at the outboard edge of the plasma. The localization widths η and Δ depend strongly on the magnitude of m, as shown in Fig. 1. The localization radius is rather insensitive to the mode numbers but depends on the density parameter σ. The frequency splitting due to consecutive poloidal and toroidal mode numbers depends on the magnitude of the mode numbers. The difference can be illustrated by a simplified analysis neglecting the Hall term: For low poloidal and toroidal mode numbers ($n \simeq 0$), Eq. (9) yields the solution $\omega \simeq mv_A/(\kappa\rho\omega_{ci})$. Then, the frequency splitting because the discrete m-numbers will be proportional equal to $\Delta f_m = v_A/(2\pi\kappa\rho_0)$ and for the NSTX experimental parameters $\Delta f_m \simeq 200$ kHz, which is larger than the experimentally observed splitting between the peaks (120 kHz). For moderate n, such as $K_n \simeq 0$, the solution of Eq. (9) is approximately $\omega/\omega_{ci} \simeq nv_A/\sqrt{n^2v_A^2 + R^2\omega_{ci}^2}$, and the frequency splitting due to n can be estimated to be $\Delta f_n = v_A/(2\pi R)$. For NSTX experimental parameters $\Delta f_n \simeq 95$ kHz (about 20% lower than the observed splitting). For moderate n, the frequency splitting due to m is negligible.

Figure 1: *Contour plots showing the radial and poloidal variation of the real part of* $\hat{B}$ *for* $R_0/a = 1.3$, $\sigma = 1$, *and mode numbers* $n = 10$, $m = 20$ *(first),* $n = 10$, $m = 10$ *(second),* $n = 10$, $m = -5$ *(third) and* $n = 0$, $m = -5$ *(fourth). The corresponding eigenfrequencies are* $\omega = 0.44\,\omega_{ci}$, $\omega = 0.44\,\omega_{ci}$, $\omega = 1.0\,\omega_{ci}$ *and* $\omega = 0.71\,\omega_{ci}$. *Note that the solutions are better localized for higher* m.



Because of the Hall term, the solutions are not symmetrical with respect to the sign of the poloidal mode number. The difference is most significant for low toroidal mode numbers, in which case there does not always exist localized solutions for positive poloidal mode numbers. The existence of localized solutions is affected by the sign of the Hall term, which depends on the radial derivative of the magnetic field.

The present analysis shows that the FME are edge-localized both poloidally and radially in the outboard edge of the plasma. The solutions for higher mode numbers are better localized and the frequency splitting is closer to the experimentally observed one. For $\sigma = 0.5$, the eigenmode structure is radially wider and is localized closer to the edge, compared to $\sigma = 1$. For relating with the SICE experimental data, apart from wave localization, also the resonance condition needs to be satisfied and a positive growth rate has to be found. These issues are out of the scope of this paper.

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