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Computational Models of Toroidal Steady-State Plasmas for the Purpose of Stability Investigations

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Introduction

The determination of steady states of ideal plasmas with mass flow is an intrinsically nonlinear problem in toroidal geometry. Although stationary equilibria with mixed (poloidal and toroidal) flow in magnetic field configurations of fusion devices have already been studied earlier [1, 2, 3], there are only few investigations [4, 5] which consider the impact of nonlinear, not purely toroidal equilibrium flow on plasma stability. As an indispensable prerequisite for this purpose we have calculated highly accurate numerical solutions of the mixed-flow equilibrium problem for the toroidal geometry of ASDEX Upgrade and present analytical solution expressions for large-aspect ratio configurations with circular flux surfaces.

Theory

Ideal plasma equilibria without flow are determined by prescribing distributions for poloidal current and plasma pressure as functions of the poloidal flux Ψ and then, after imposing suitable boundary conditions, solving the usual partial differential equation for Ψ. For stationary equilibria with poloidal and toroidal flow this determination requires five such flux functions: Poloidal mass flux Ψ_M, poloidal current J_M, dynamic free enthalpy G_M, electric potential Φ_M and temperature T. Here a partial differential equation must be solved together with two nonlinear algebraic equations for mass density ρ and poloidal current J:

$$\nabla\cdot\Big\{\Big(1-\frac{\mu_0{\Psi'_M}^2}{\rho}\Big)\frac{\nabla\Psi}{R^2}\Big\}+\frac{\mu_0^2 J J'_M}{R^2} \\ +4\pi^2\mu_0\big(\rho({G_M}'-\frac{\partial G}{\partial T}T')+\boldsymbol{v}\cdot\boldsymbol{B}\,{\Psi_M}''+C_M{\Phi_M}''\big)=0 \tag{1}$$

$$\frac{B^2{\Psi'_M}^2}{2\rho^2}-2\pi^2R^2{\Phi'_M}^2+G(T,\rho kT/m)-G_M=0 \tag{2}$$

$$-4\pi^2R^2\Psi'_M{\Phi_M}'+(1-\frac{\mu_0{\Psi'_M}^2}{\rho})J-J_M=0 \tag{3}$$

where $'=d/d\Psi$ and

$$\boldsymbol{v}\cdot\boldsymbol{B}=\frac{\Psi'_M|\nabla\Psi|^2+\mu_0 J C_M}{4\pi^2\rho R^2},\;\; C_M=\mu_0 J{\Psi_M}'+4\pi^2R^2\rho{\Phi_M}',\;\; G(T,p)=\frac{kT}{m}\ln\Big(\frac{p}{p_M}\Big) \tag{4}$$

G is the specific Gibbs free enthalpy and R is the distance from the axis of symmetry. All other quantities have their usual meaning. After solving equations (1) - (3) magnetic field $\boldsymbol{B}$ and momentum density $\rho\boldsymbol{v}$ are given by

$$\boldsymbol{B}=\frac{1}{2\pi}\Big(\nabla\Psi\times\nabla\varphi+\mu_0 J\nabla\varphi\Big) \tag{5}$$

$$\rho\boldsymbol{v}=\frac{1}{2\pi}\Big(\nabla\Psi_M\times\nabla\varphi+C_M\nabla\varphi\Big) \tag{6}$$

We remark that four of the above-quoted five flux functions can be replaced by the

physically more apparent quantities J, $p = \rho kT/m$ and the magnetic Mach numbers $m_p = \sqrt{\mu_0\rho}v_p/B_p$ and $m_t = \sqrt{\mu_0\rho}v_t/B_t$ prescribed along a suitable profile specification line leading from the magnetic axis to the plasma boundary.
For circular flux surfaces and zeroth order in inverse aspect-ratio $\epsilon = a/R_0$ the partial differential equation (1) reads

$$\Delta\Psi + \frac{\mu_0^2}{2}\frac{dJ^2}{d\Psi} + 4\pi^2\mu_0 R_0^2\frac{dp}{d\Psi} + \frac{\mu_0\rho v_p^2}{B_p^2}\left(2\pi^2 R_0^2\frac{dB_p^2}{d\Psi} - \Delta\Psi\right) = 0 \tag{7}$$

As shown in [6] this equation for the poloidal magnetic flux Ψ can be integrated and gives for the poloidal magnetic field and for Ψ

$$B_p^2 = \langle\, B_t^2\,\rangle - B_t^2 + 2\mu_0(\langle p\rangle - p) + \mu_0\langle\rho v_p^2\rangle, \quad \Psi(R,z) = \pi a R_0\int_{\mathrm{v(R,z)}}^1 B_p(\mathrm{v}')\frac{\mathrm{dv}'}{\sqrt{\mathrm{v}'}} \tag{8}$$

which is the Bennet-pinch relation generalized for the inclusion of plasma flow. Here

$$\langle\, f\,\rangle(\mathrm{v}) \equiv \frac{1}{\mathrm{v}}\int_0^{\mathrm{v}} \mathrm{f(v')dv'}, \qquad \mathrm{v} = \frac{\mathrm{V}}{\mathrm{V_b}} = \frac{\mathrm{r^2}}{\mathrm{a^2}} \tag{9}$$

for any quantity f depending on the normalized volume v. V is the volume of the circular magnetic surface under consideration and a is the small radius so that $V_b = V(a)$ is the total volume of the plasma. In what follows we conceive all zeroth order quantities as function of v. As in the general case we consider the problem in terms of the magnetic Mach number distributions m_p and m_t. However, because of the relation $\mu_0\langle\,\rho\mathrm{v_p^2}\,\rangle = \langle\, B_p^2 m_p^2\,\rangle$, in this case the integration of equation (7) must be reconsidered in order to obtain B_p^2 explicitly. It turns out that it is given by the solution of the following ordinary differential equation

$$\mathrm{v}\frac{\mathrm{dB_p^2}}{\mathrm{dv}} + (1-\mathrm{m_p^2})\mathrm{B_p^2} + \mathrm{v}\frac{\mathrm{d(B_t^2} + 2\mu_0\mathrm{p})}{\mathrm{dv}} = 0 \tag{10}$$

Expanding now (1) to first order in ϵ using shifted-circle coordinates (r,α) it turns out that all terms are proportional to $\cos\alpha$. Their sum must vanish in equilibrium. This can be accomplished by chosing a displacement function Δ such that

$$\frac{d\Delta}{d\mathrm{v}} = -\frac{\epsilon a}{2}\frac{1}{D_0}(\beta_p + \frac{l_i}{2} + l_m), \quad D_0 = \frac{(1-m_p^2)(m_c^2 - m_p^2)}{m_c^2(1+m_p^2M_p^2) - m_p^2}, \quad m_c^2 \equiv \frac{\mu_0 p}{\mu_0 p + B^2} \tag{11}$$

with m_c being the critical poloidal Mach number for which the partial diffential equation (7) would leave its first elliptic regime. $M_p = v_p/c_s$ is the acoustic poloidal Mach number, where $c_s^2 = kT/m$. Further

$$\beta_p = \frac{2\mu_0(\langle\, p\,\rangle - p)}{B_p^2}, \qquad l_i = \frac{\langle\, B_p^2\,\rangle}{B_p^2} \tag{12}$$

are the poloidal beta and the coefficient of the inner inductance of the plasma. l_m is the inductance contribution caused by flow and given by

$$l_m = \frac{1}{B_p^2}\Big\{\langle\, m_t^2 B_t^2\,\rangle - m_t^2 B_t^2 + \frac{1}{2}\langle\, m_p^2 B_p^2\,\rangle - m_p^2 B_p^2\Big\}$$
$$-m_p^2(1-m_c^2)\frac{(m_t - m_p)^2 B_t^2/B^2 + m_p^2(1 - m_p^2 B_p^2/B^2 - m_t^2 B_t^2/B^2)}{m_c^2(1+m_p^2 M_p^2) - m_p^2} \tag{13}$$

After integration of (11) with the boundary condition $\Delta(\mathrm{v}=1) = 0$ we see that the radial displacement of the magnetic surfaces is of the functional type $\Delta = \Delta(m_p, m_t, \mathrm{v})$ and by its flow dependence is responsible for the first-order variation of mass density and poloidal current on magnetic surfaces. We advance the notation for ρ and J for the inclusion of first order contributions and write

$$\rho = \rho^{(0)}(\mathrm{v}) + \rho^{(1)}(\mathrm{m_p, m_t, v})\cos\alpha, \quad \mathrm{J} = \mathrm{J}^{(0)}(\mathrm{v}) + \mathrm{J}^{(1)}(\mathrm{m_p, m_t, v})\cos\alpha \tag{14}$$

For the determination of $\rho^{(1)}$ and $J^{(1)}$ equations (2) and (3) must be solved. For brevity we omit here the lengthy explicit expressions obtained for J and ρ.

Calculations

We have calculated numerical solutions of (1) by solving the free-boundary Poisson problems

$$R^2\nabla\cdot(\nabla\Psi^{(n)}/R^2) + 2\pi\mu_0 R j_t(R, z, \boldsymbol{P}_s(\Psi^{(n-1)})) = 0 \tag{15}$$

in $n \leq 20$ iteration steps starting with a static solution $\Psi^{(0)}$ determined in advance. Here j_t is the toroidal component of the current density extracted from (1) and $\boldsymbol{P}_s$ comprises given distributions of J, ρ, T, m_p and m_t on a profile specification line leading from the magnetic axis to the plasma boundary. In each iteration

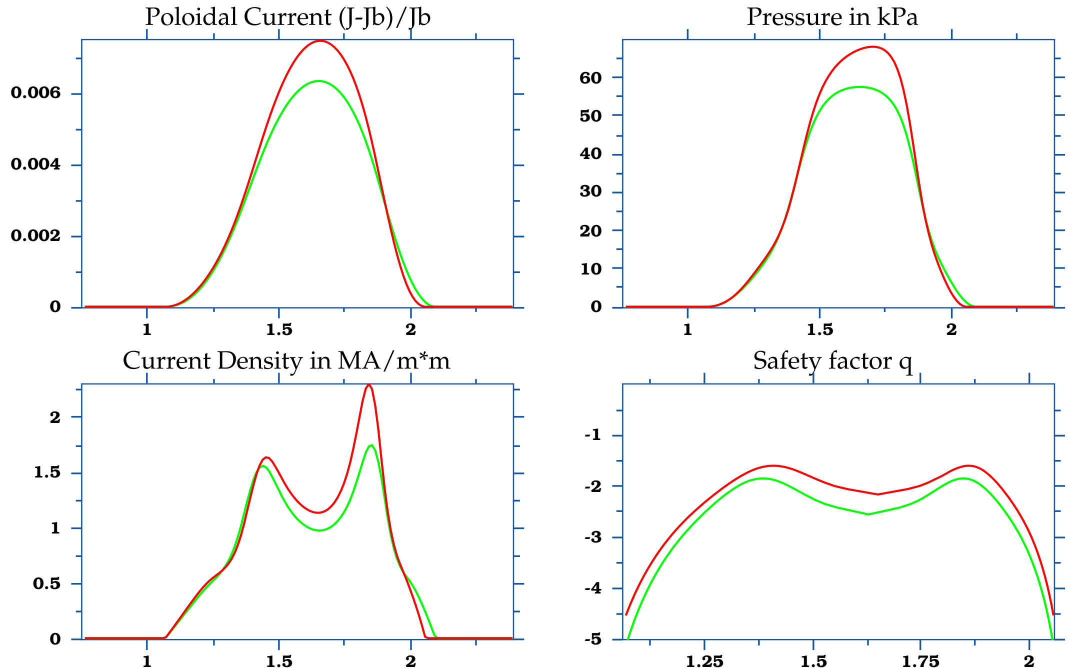

Figure 1: Equatorial-plane equilibrium profiles of a static equilibrium (green curves) with superposed poloidal ($m_p = 0.01$) and toroidal ($m_t = 0.1$) flow (red) as functions of the distance R in m from the axis of symmetry.

also the implicit equations (2) and (3) for mass density and poloidal current are solved using a continuation method. The profiles $\boldsymbol{P}_s$ are chosen such that for $m_p = m_t = 0$ a well-defined static equilibrium is obtained which was previously used for stability investigations with purely toroidal flow [7]. Starting with calculations with purely toroidal flow the poloidal Mach number was successively increased until its value was close to the limit set by the characteristic determinant for being positive. In this way the requirements for a boundary value problem in the first elliptic regime were satisfied everywhere in the plasma region. All calculations were done for a fixed magnetic axis position and given values for plasma current and beta-poloidal. For both equilibria the plasma limiting point is on the high-field side (Fig. 1). Inspecting the profiles it can be seen that the plasma with flow is radially more compressed and that the point of maximum pressure is displaced with respect to the magnetic axis by about 4 cm in radial direction. As an application of the large-aspect-ratio analytical theory with shifted circular flux surfaces we have done a model calculation for $A = 1/\epsilon = 6$. Fig. 2 shows the poloidal contours of constant mass density and constant poloidal current obtained by the evaluation of the equations (14). They illustrate the fact that poloidal flow is connected with local currents $\boldsymbol{j}\cdot\nabla\Psi \neq 0$ flowing across magnetic surfaces which might release an additional channel to instability.

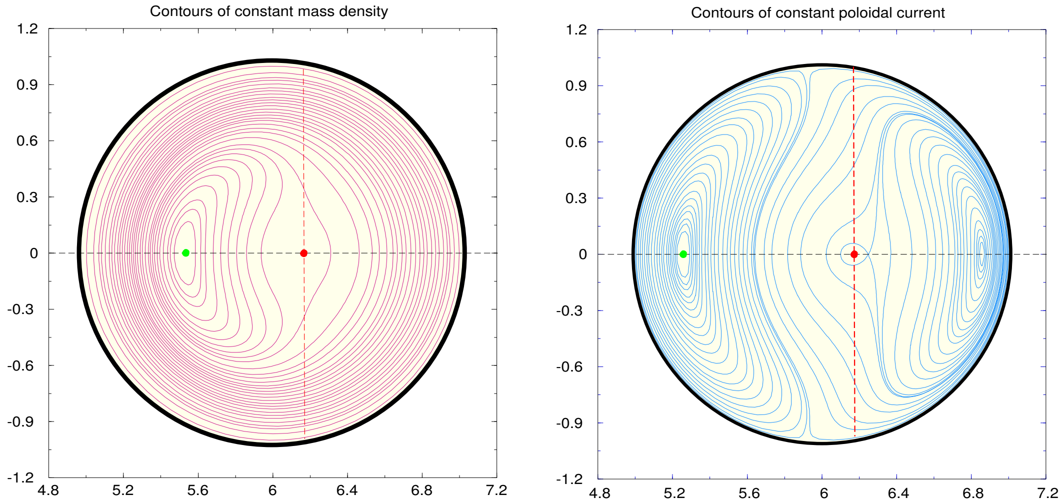

Figure 2: Contours of constant mass density (left) and contours of constant poloidal current (right). The profile specification line (red, dotted) passes the magnetic axis at maximum values of mass density and poloidal current, respectively.

Summary and Conclusions

The computational procedures and algorithms for the calculation of stationary states of rotating plasmas as described above were implemented as an extension of the DIVA equilibrium program which was used also previously [7] as equilibrium-data providing tool for CASTOR-code stability analysis on double-tearing modes. Since DIVA has an object-oriented design and is organized in form of computation classes with object persistence (in form of XDR files), stability codes such as CASTOR can easily access data and services of objects of these classes. We are now extending an already existing DIVA-CASTOR interface for the inclusion of both poloidal and toroidal mass flow not only into equilibrium but also into stability investigations.

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