

Analysis of a cyclotron maser instability with application to space and laboratory plasmas

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Introduction

When a beam of electrons moves into a converging magnetic field, the velocity distribution function takes on a horseshoe shape as a result of conservation of magnetic moment and energy. A few years ago it was pointed out that such a distribution is unstable to a cyclotron maser type of instability and it was suggested that this instability might be the source of auroral kilometric radiation [1] and also of emission from certain types of star [2]. The theoretical analysis presented in the earlier papers is somewhat approximate and assumes an infinite uniform plasma. We discuss in this paper a more exact theory in cylindrical geometry, showing how to derive a dispersion relation for TE modes. This geometry is relevant to the auroral problem, where the emission is observed to be generated in roughly cylindrical low density cavities. It is also relevant to a laboratory experiment which we are currently carrying out (see [3]). The latter exploits the fact that the mechanism only depends on dimensionless ratios like the ratio of the wave frequency to the cyclotron and plasma frequencies and the factor by which the magnetic field varies along the path of the electron beam. Thus it is possible to scale it to laboratory size and to GHz and above frequencies.

In this paper we consider how the growth rate depends on the mode structure and look at the effect of different radial distributions of the driving electron beam. We show that, in line with our earlier analysis and simulations [3], the growth rate of the instability is large - sufficient to give high amplification on the scale of the experiment.

Distribution function and dielectric tensor

An electron beam moving into a converging magnetic field changes its momentum distribution. For a drifting Maxwellian initial distribution the laws of conservation of magnetic moment and particle energy mean that we obtain the transformed distribution with a shape of a horseshoe in velocity space [1],

$$f(v_{\parallel}, v_{\perp}) = Ae^{-\frac{m}{2T} \left[\left(\sqrt{v_{\parallel}^2 + (1-B/B_0)v_{\perp}^2} - v_0 \right)^2 + B/B_0 v_{\perp}^2 \right]}, \quad (1)$$

see Fig.1. The horseshoe distribution so formed gives rise to a cyclotron maser instability which is considered here as a source of radiation.

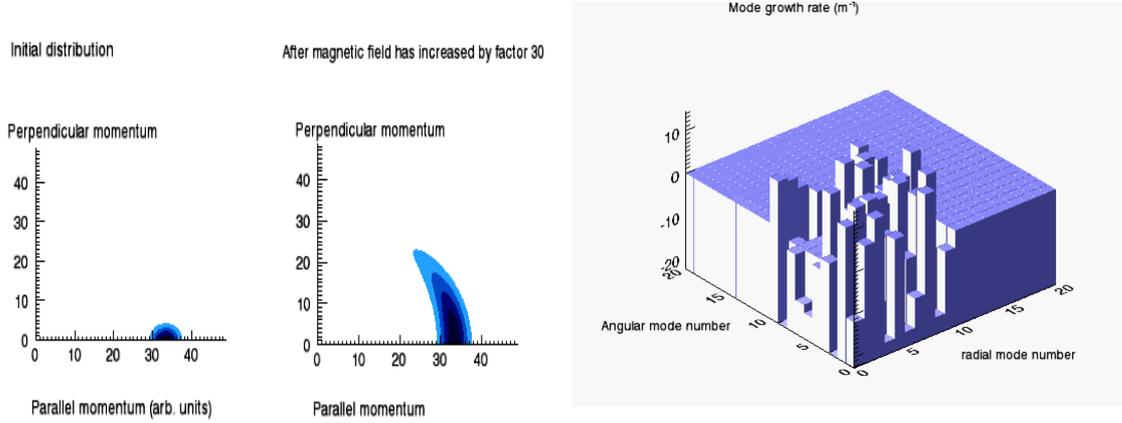


Fig.1. Horseshoe distribution formation.

Fig.2. Growth rate of different modes for

$R=4.0\text{cm}$, $R_0=0.4\text{cm}$, $R_1=2.1\text{cm}$,
 $\omega / \Omega = 0.998527$, $T=50\text{eV}$,
 beam energy =500keV.

The dielectric tensor of this plasma for a wave frequency near the cyclotron frequency, including relativistic effects which have an important effect on the wave-particle resonance condition, is [4]

$$\hat{K}_{\perp} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \left(1 - \frac{P_e^2}{\omega^2} \right) \hat{I} - \frac{P_e^2}{\omega^2} \int \left\{ \frac{\hat{T}}{k_z v_{\parallel} - \omega + \Omega \sqrt{1 - (v_{\parallel}^2 + v_{\perp}^2)/c^2}} \right. \\ \left. \left(-\frac{n\Omega}{v_{\perp}} \frac{f}{v_{\perp}} + k_z \frac{f}{v_{\parallel}} \right) \frac{1}{n_0} \right\} d\vec{v}, \quad (2)$$

where the matrix \hat{T} can be simplified for small Larmor radius as $\hat{T} = \frac{1}{4} v_{\perp}^2 \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$, P_e and Ω are the plasma and cyclotron frequencies (without relativistic corrections), respectively, and f is determined by (1). The integral in (2) can be reduced to a single integral after changing the coordinates and using residues of the function.

Theory in cylindrical geometry

We consider an annular electron beam, forming a ring $R_0 \leq r \leq R_1$ of plasma inside a transverse cross section of a circular metallic tube with its radius equal to R . We consider a perfectly conducting boundary, though other boundary conditions could be imposed.

To investigate the radiation properties of the horseshoe distribution instability we look for the growth rate of different modes, determined as the imaginary part of a complex-value k_z in the field propagator $e^{-i\omega t + ik_z z}$. The frequency is chosen to be just below the cyclotron frequency, in the range where the instability is expected to occur. The problem is reduced now to solving Maxwell's equations in cylindrical coordinates for the transverse field components in a circular waveguide partially filled with anisotropic medium, whose properties are determined by the dielectric tensor (2).

We restrict consideration here to TE modes with $E_z(r, \varphi) = 0$. Since the plasma couples radial and azimuthal field components we seek a solution in a more general form than for conventional TE modes in isotropic waveguides. We search for a solution inside the plasma region in the form

$$E_r(r, \varphi) = \frac{J_n(\beta_{pl} r / R_1)}{r} (A_1 \cos(n\varphi) + A_2 \sin(n\varphi)) + J_n'(\beta_{pl} r / R_1) (B_1 \cos(n\varphi) + B_2 \sin(n\varphi)),$$

$$E_\varphi(r, \varphi) = \frac{J_n(\beta_{pl} r / R_1)}{r} (C_1 \cos(n\varphi) + C_2 \sin(n\varphi)) + J_n'(\beta_{pl} r / R_1) (D_1 \cos(n\varphi) + D_2 \sin(n\varphi)),$$

where J_n and J_n' are the Bessel functions of the first kind and their derivatives. This form is appropriate to a cylindrical beam on the axis. For the annular beam we add extra terms involving the Bessel functions of the second kind. So, the solution we seek has each of the transverse components as a sum of those for both TE and TM modes in an isotropic waveguide, but with different coefficients for each component, which makes the sum not the sum of the vectors \vec{E}^{TE} and \vec{E}^{TM} , but what we call an anisotropic sum,

$$\begin{cases} E_r = a E_{r_isotropic}^{TE} + b E_{r_isotropic}^{TM} \\ E_\varphi = c E_{\varphi_isotropic}^{TE} + d E_{\varphi_isotropic}^{TM} \end{cases} .$$

This form of solution allows us to obtain an exact but simple dispersion relation for the plasma region,

$$\left(\frac{\omega^2}{c^2}k_{11} - k_z^2\right)\left(\frac{\beta_{pl}^2}{R_1^2} - \frac{\omega^2}{c^2}k_{11} + k_z^2\right) = \left(\frac{\omega^2}{c^2}\right)^2 k_{12}^2 \quad (3)$$

where k_{11} and k_{12} are the elements of the dielectric tensor (2) and depend on the radiation frequency ω . The equation for finding β_{pl} can be obtained by applying the boundary conditions, with the field in the vacuum region being expressed in a conventional way, resulting in

$$\frac{nk_{11}^2 J_n(\beta_{pl}) - \beta_{pl} J_n'(\beta_{pl})}{nk_{12}^2 J_n(\beta_{pl}) - \beta_{pl} J_n'(\beta_{pl})} = \frac{nk_{11}^2 Y_n(\beta_{pl}R_0/R_1) - \beta_{pl}R_0/R_1 Y_n'(\beta_{pl}R_0/R_1)}{nk_{12}^2 Y_n(\beta_{pl}R_0/R_1) - \beta_{pl}R_0/R_1 Y_n'(\beta_{pl}R_0/R_1)}. \quad (4)$$

Equations (3) and (4) provide an accurate and very fast way to analyze the horseshoe instability radiation. A representative example showing the growth rate of different modes is shown in Fig.2. The growth length of 5-10 cm for the fastest growing modes is in good agreement with the typical growth length seen in particle-in-cell simulations of this instability [3].

Conclusions

Our analysis indicates that the horseshoe distribution produces a rapidly growing instability at frequencies just below the electron cyclotron frequency. Depending on the proportions of the cavity, a variety of different modes may be unstable, with large growth rates. Growth rates are greatest for the modes with smallest k_z , in line with expectations that the instability is favoured by near perpendicular propagation. Analysis for different geometries of the beam including the situations where it fills a central region or an annular region shows that in all cases strong instability is possible. It suggests the current model is closer to explaining such phenomenon as auroral kilometric radiation than the loss-cone instability which has been proposed in the past.

Future work will include comparing the predictions of these calculations with the Strathclyde experiment and with simulation codes [3].

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