*31st EPS Conference on Plasma Phys. London, 28 June - 2 July 2004 ECA Vol.**28G**, P-4.221 (2004)*

Invariant Wave Dispersion Relation and Semiclassical Conservation Laws

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Introduction. Semiclassical asymptotic techniques, in particular, the geometrical optics (GO) method [1, 2] and the wave kinetic equation [3], constitute the standard tools to describe the propagation of waves in slowly space- and time-varying media, as they allow to reduce the relevant (integro-differential) wave equation to a set of (tractable) transport equations. More specifically, with reference to a generic wave equation $\big(\hat{\mathbf{D}}\cdot\boldsymbol{\psi}\big)(\mathbf{r},t)=0$ for the multi-component wavefield $\boldsymbol{\psi}(\mathbf{r},t)$, the matrix of operators $\hat{\mathbf{D}}$ corresponds to a set of (real-valued) Hamiltonian functions $\Omega^{(\sigma)}(\mathbf{k},\mathbf{r},t)$ (the eigenfrequency of the mode σ) determining a Hamiltonian system in the "position-wavevector" ($\mathbf{r}$-$\mathbf{k}$) phase space. Such functions are conveniently determined on solving the local dispersion relation $D^{(\sigma)}(\mathbf{k},\omega;\mathbf{r},t)=0$ in terms of ω, with $D^{(\sigma)}$ the eigenvalues of the *Weyl symbol* $\mathbf{D}(\mathbf{k},\omega;\mathbf{r},t)$ of the operator $\hat{\mathbf{D}}$, the corresponding (unit) eigenvectors $\mathbf{e}^{(\sigma)}(\mathbf{k},\omega;\mathbf{r},t)$ being related to the polarization of the considered eigenmode [3]. To the extent that mode conversion can be neglected, each mode propagates independently from the others and the physical quantities characterizing the wavefield are obtained from the dynamics of the corresponding Hamiltonian system, to lowest order in the small parameter $\delta\sim|\mathbf{k}L|^{-1}\sim|\omega T|^{-1}(\ll 1)$, with L (T) the scale length (time) of the space- (time-) variations of $\mathbf{D}$. For instance, in the GO method, the leading order part of the phase, i.e., the (real) eikonal function $S^{(\sigma)}(\mathbf{r},t)$, satisfies the Hamilton-Jacobi equation $\partial_t S^{(\sigma)}+\Omega^{(\sigma)}(\nabla S^{(\sigma)},\mathbf{r},t)=0$, whereas the wavefield (squared) amplitude $|A^{(\sigma)}(\mathbf{r},t)|^2$ is obtained from the continuity equation for the quantity $q_D^{(\sigma)}=\partial_\omega D^{(\sigma)}|A^{(\sigma)}|^2$, namely,

$$\partial_t q_D^{(\sigma)}(\mathbf{r},t)+\nabla\cdot\Big(\mathbf{v}_g^{(\sigma)}(\mathbf{r},t)q_D^{(\sigma)}(\mathbf{r},t)\Big)=-2\gamma^{(\sigma)}(\mathbf{r},t)\;q_D^{(\sigma)}(\mathbf{r},t), \tag{1}$$

where the streamlines of the group velocity $\mathbf{v}_g^{(\sigma)}(\mathbf{r},t)$ are the projection onto the $\mathbf{r}$-space of the Hamiltonian trajectories in the $\mathbf{r}$-$\mathbf{k}$ phase space, the coefficient $\gamma^{(\sigma)}\equiv(\partial_\omega D^{(\sigma)})^{-1}\;\mathbf{e}^{(\sigma)*}\cdot\mathbf{D}_A\cdot\mathbf{e}^{(\sigma)}\;\big(=O(\delta)\big)$, i.e., the projection of the anti-Hermitian part of $\mathbf{D}$ along the mode polarization $\mathbf{e}^{(\sigma)}$, accounts for dissipation and/or instabilities and all functions of $(\mathbf{k},\omega)$ are evaluated at $\mathbf{k}=\nabla S^{(\sigma)}$, $\omega=-\partial_t S^{(\sigma)}$. It is worth noting that the physical meaning of the quantity $q_D^{(\sigma)}$ depends on $D^{(\sigma)}$, e.g., it represents the wave energy density [1, 2] or the wave action density [3] depending on how $D^{(\sigma)}$ is normalized.

On the other hand, the wave kinetic formalism is based on the the Wigner-Weyl phese-space representation of operators [3], for which the wavefield intensity amounts to $|\boldsymbol{\psi}(\mathbf{r},t)|^2=\sum_\sigma\int\frac{d^3kd\omega}{(2\pi)^4}W^{(\sigma)}(\mathbf{k},\omega;\mathbf{r},t)$, where the spectral (or Wigner) function $W^{(\sigma)}$ represents the "distribution of wave quanta" in the mode σ in the $(\mathbf{k},\omega;\mathbf{r},t)$-space such

that $W^{(\sigma)}(\mathbf{k},\omega;\mathbf{r},t) = 2\pi Q_D^{(\sigma)}(\mathbf{k},\mathbf{r},t)\delta\big(D^{(\sigma)}(\mathbf{k},\omega;\mathbf{r},t)\big)$. The (unknown) function $Q_D^{(\sigma)}$ satisfies a Boltzmann-like equation in the **r**-**k** phase space referred to as the *wave kinetic equation*, namely,

$$\partial_t Q_D^{(\sigma)} + \big\{Q_D^{(\sigma)}, \Omega^{(\sigma)}\big\} = -2\gamma^{(\sigma)}\ Q_D^{(\sigma)} \tag{2}$$

where $\{\cdot,\cdot\}$ denotes the Poisson brackets in the **r**-**k** phase space and $\gamma^{(\sigma)}$ is the same coefficient entering the GO transport equation (1) with $\omega = \Omega^{(\sigma)}(\mathbf{k},\mathbf{r},t)$ and $\mathbf{k}$ an independent variable. Analogously to $q_D^{(\sigma)}$, the quantity $Q_D^{(\sigma)}$ depends on $D^{(\sigma)}$ and, in particular, amounts to the wave action density in phase space for a suitable normalization.

Let us note that, on making use of the Weyl symbol, the Hamiltonian structure underlying the propagation of the wavefield is the same in both the GO method and the wave kinetic formalism. However, this is no longer true if other formulations of the GO method are considered, e.g., the *non-symmetrized* formulation for which the Weyl symbol is replaced by the plane-wave response tensor [4]; moreover, the operator $\hat{\mathbf{D}}$ can be expressed in many equivalent forms each one related to the other by multiplication of an invertible operator $\hat{E}$, so that many different forms of the dispersion function are obtained.

In this work, a modification of the standard GO method and wave kinetic formalism is proposed for which the local dispersion relationship is the same in every GO formulations as well as in the wave kinetic formalism and does not depend on the form of $\hat{\mathbf{D}}$ so that it is an *intrisic* property of the considered physical system.

The intrinsic Hamiltonian structure. Let us begin on noting that the operator $\hat{\mathbf{D}}$ is *pseudodifferential* [3-6], i.e., it can be written as an integral operator with the corresponding kernel expressed by an oscillatory (Fourier) integral, namely,

$$\big(\hat{\mathbf{D}}\cdot\boldsymbol{\psi}\big)(x) = \frac{1}{(2\pi)^4}\int e^{ik\cdot(x-x')}\ \mathbf{d}(k,x,x')\cdot\boldsymbol{\psi}(x')\ d^4x'\ d^4k, \tag{3}$$

where $x = (\mathbf{r},ct)$ and $k = (\mathbf{k},-\omega/c)$ for simplicity. In equation (3) the amplitude $\mathbf{d}(k,z)$, $z=(x,x')$, is a smooth (matrix-valued) function whose components belong to the class of functions referred to as *symbols* [5, 6]. In particular, the specific asymptotic properties of such functions allow to reduce the symbol $\mathbf{d}(k,z)$, which depends on 12 variables, to a function of 8 variables by means of an asymptotic series expansion. More specifically, $\hat{\mathbf{D}}$ is asymptotically equivalent (in the sense of microlocal analysis [6]) to the pseudodifferential operator with symbol

$$\mathbf{d}^{(q,p)}\big(k,qx+px'\big) \sim \sum_{j=0}^{+\infty}\mathbf{d}_j^{(q,p)}\big(k,qx+px'\big), \tag{4}$$

which depends on x and x' only through the (convex) combination $r = qx+px'$ with $q+p=1$, thus effectively reducing the number of variables. For each choice of q and

p, one gets a different asymptotic representation $\mathbf{d}^{(q,p)}$ of the considered operator in terms of an asymptotic series of (decreasing order) symbols $\mathbf{d}_j^{(q,p)}$. In particular, for $q = 1$, $p = 0$ and $q = p = \frac{1}{2}$, respectively, the non-symmetrized [4] and the Weyl (or symmetrized [3, 4]) representations are retrieved. It is essential that the leading order term $\mathbf{d}_0^{(q,p)}(k,r) = \mathbf{d}(k,r,r) \equiv \mathbf{D}^0(k,r)$ in the series (4) does not depend on the adopted representation, i.e., it is an *intrinsic property of the operator*, to be referred to as *principal symbol*.

On making use of the asymptotic series (4) in the standard GO asymptotics (applied to any (q,p)-representation) as well as in the Wigner-Weyl formalism, one has that the dispersion function $D^{(\sigma)}(k,x)$ is *properly* determined as an eigenvalue of the principal symbol $\mathbf{D}^0(k,x)$ which is assumed to be Hermitian, the corresponding eigenvector being the *proper* polarization (unit) vector $\mathbf{e}^{(\sigma)}(k,x)$. On solving the dispersion equation $D^{(\sigma)} = 0$ for the frequency ω, one gets the Hamiltonian $\Omega^{(\sigma)}(\mathbf{k},\mathbf{r},t)$ which is unique in view of the uniqueness of the principal symbol. Therefore, the Hamiltonian structure thus obtained appears to be a fundamental property of the considered physical system and it is the same in both the GO method and the wave kinetic formalism. As for the absorption coefficient $\gamma^{(\sigma)}$ entering both the GO transport equation (1) and the wave kinetic equation (2), one has $\gamma^{(\sigma)} \equiv (\partial_\omega D^{(\sigma)})^{-1}\ \mathbf{e}^{(\sigma)*} \cdot \mathbf{D}_A^1 \cdot \mathbf{e}^{(\sigma)}$ with $\mathbf{D}_A^1$ the anti-Hermitian part of the *subprincipal Weyl symbol* $\mathbf{d}_1^{(\frac{1}{2},\frac{1}{2})} \equiv \mathbf{D}^1$, such an expression for $\gamma^{(\sigma)}$ being also independent of the adopted formulation.

Let us now examine how the dispersion function and the absorption coefficient vary if the equivalent form $\hat{\mathbf{D}}' = \hat{E} \cdot \hat{\mathbf{D}}$ of the operator is considered (instead of $\hat{\mathbf{D}}$). It is readily obtained, e.g., on exploiting the Weyl calculus [3], that the principal symbol and the subprincipal Weyl symbol of the new operator are, respectively, $E^0\mathbf{D}^0$ and $E^0\mathbf{D}^1 + E^1\mathbf{D}^0 + \frac{i}{2}\{E^0, \mathbf{D}^0\}_{x\text{-}k}$ where the superscript "0" and "1" refer to the principal and subprincipal Weyl symbol, respectively, whereas $\{\cdot,\cdot\}_{x\text{-}k}$ denotes the Poisson brackets in the 8-dimensional x-k phase space. From a semiclassical viewpoint, the invertibility of the operator $\hat{E}$ amounts to requiring that $E^0 \neq 0$ [6]; furthermore, let us assume that E^0 is real-valued. Under these conditions the dispersion functions are $E^0 D^{(\sigma)}$ so that the solution of the dispersion equation $E^0 D^{(\sigma)} = 0$ yields the same eigenfrequency as $D^{(\sigma)} = 0$ since $E^0 \neq 0$. One can conclude that the Hamiltonian structure determined by the principal symbol is invariant also with respect to the different forms of the operator $\hat{\mathbf{D}}$. On the other hand, the absorption coefficient $\gamma^{(\sigma)}$ changes according to

$$\gamma^{(\sigma)} \longrightarrow \gamma^{(\sigma)\prime} = \gamma^{(\sigma)} - \frac{1}{2E^0}\frac{dE^0}{dt}, \tag{5}$$

where d/dt denotes the derivative along the Hamiltonian trajectories. However, the corresponding GO transport equation and wave kinetic equation reduce exactly to equations (1) and (2), respectively, as the transported quantities, namely, $q_D^{(\sigma)}$ and

$Q_D^{(\sigma)}$, depend on the dispersion function $D^{(\sigma)}$ so that

$$q_D^{(\sigma)} \longrightarrow q_{D'}^{(\sigma)} = E^0(\mathbf{k},\omega;\mathbf{r},t)\; q_D^{(\sigma)}, \quad \text{with } \mathbf{k} = \nabla S^{(\sigma)},\; \omega = -\partial_t S^{(\sigma)}, \tag{6a}$$

$$Q_D^{(\sigma)} \longrightarrow Q_{D'}^{(\sigma)} = E^0(\mathbf{k},\omega;\mathbf{r},t)\; Q_D^{(\sigma)}, \quad \text{with } \omega = \Omega^{(\sigma)}(\mathbf{k},\mathbf{r},t). \tag{6b}$$

The transformations (5) and (6) show that different *but equivalent* forms of both the GO transport and wave kinetic equations exist along with different transported quantities, depending on the form of the operator $\hat{\mathbf{D}}$.

Discussion. On making use of the asymptotic series (4), one has that the eigenfrequency of a (propagating) mode is obtained from the *principal symbol* of the relevant pseudodifferential operator, the corresponding Hamiltonian structure being an *intrinsic* property of the considered wavefield. Moreover, the GO transport and wave kinetic equations are uniquely determined in the sense that the different forms of such equations are all equivalent.

Despite its formal definition, the concept of principal symbol is readily appreciated on considering a specific case, e.g., the equation $\hat{D}\Phi \equiv \nabla\cdot(\boldsymbol{\varepsilon}\cdot\nabla\Phi) = 0$ relevant to electrostatic waves with Φ the scalar potential and $\boldsymbol{\varepsilon} = \left(\varepsilon^i{}_j(\mathbf{r},\omega)\right)$ the dielectric tensor [7]. The plane-wave response of the (scalar) operator $\hat{D}$ amounts to $d^{(1,0)} = -\frac{1}{2}\varepsilon^{ij}k_ik_j + ik_j\frac{\partial}{\partial x^i}\left(\delta^{jl}\varepsilon^i{}_l + \delta^{il}\varepsilon^j{}_l\right)$, with $\varepsilon^{ij} = \delta^{jl}\varepsilon^i{}_l + \delta^{il}\varepsilon^j{}_l$, whereas its Weyl symbol is $d^{(\frac{1}{2},\frac{1}{2})} = -\frac{1}{2}\varepsilon^{ij}k_ik_j + \frac{1}{2}k_j\frac{\partial\tilde{\varepsilon}^{ij}}{\partial x^i} + \cdots$, with $\tilde{\varepsilon}^{ij} = i(\delta^{jl}\varepsilon^i{}_l - \delta^{il}\varepsilon^j{}_l)$, as obtained on the basis of $d^{(1,0)}$ [3, 4]. One should note that both the foregoing symbols are expressed in the form (4) with the same leading order term $D^0 = -\frac{1}{2}\varepsilon^{ij}k_ik_j$ which determines uniquely the dispersion relationship. On the other hand, according to the standard approach of the non-symmetrized GO formulation and of both the symmetrized GO formulation and the wave kinetic formalism, the real part of $d^{(1,0)}$ and $d^{(\frac{1}{2},\frac{1}{2})}$, respectively, determines the dispersion relation, thus yielding different results.

Let us mention, however, that the separation of the principal symbol, in general, is nonunique, in spite of being the same for every (q,p)-representation; such an additional arbitrariness should be dealt with as a *gauge transformation* [8].

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