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Wigner description of laser driven instabilities at arbitrary intensities

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Abstract. The Wigner formalism of quantum mechanics provides an alternative formulation based on the phase space representation of wave fields. This approach gives a direct connection with kinetic theory and provides a unique way to describe broadband fields.

In this work we examine the classic instabilities for laser pulses in plasmas: Raman forward scattering (RFS), modulation instability (MI), back scattering (BS) and self-focusing (SF). Following the standard procedures of the Wigner transform method, the stochastic Wigner-Moyal equation for the photons is coupled with the relativistic fluid equations for the plasma. The resulting dispersion relation holds for all values of a_0, and e recover the results from McKinstrie and Bingham [1], W. B. Mori, *et al* [2], P. Mora, *et al* [3] and C. E. Max, *et al* [4], in the limit of small a_0.

1. INTRODUCTION

Photon kinetics gives an unique way to study nonlinearities in plasmas. This formalism was first proposed by Wigner [5] to provide a the phase space representation of the wave function in quantum mechanics. It was later developed by Moyal [6]. Its first application to electromagnetic fields was made by Tappert in the 70's [7].

This formalism seems to be ideal to describe forward and backward scattering instabilities of broadband radiation sources. Furthermore the fact that photon kinetics represents fields as quasi-particles, leads to clear the physical pictures of the nonlinearities, in plasmas.

In this work, we couple the stochastic Wigner-Moyal equation for the photon density in the phase space, describing the evolution of the electromagnetic field, with the relativistic fluid equations for the electron density and momentum, describing the plasma dynamics.

We recover monochromatic (laser beam) growth rate instabilities in the under-dense and weakly relativistic limit (Raman forward scattering, self-phase modulation), we then take the non-relativistic assumption, and study the cases of RFS, BS and SF in a fully relativistic way. We will see interesting phenomena, such as the saturation of BS with a_0, and the decreasing of RFS and SF for high a_0. We will find the relativistic threshold for SF, which is compatible, for small a_0, with the standard threshold.

2. DISPERSION RELATION

We are now interested in coupling the Wigner-Moyal equation for the photons:

$$\frac{\partial \rho(\mathbf{r},\mathbf{k},t)}{\partial t} = 2\omega_k(\mathbf{r},\mathbf{k},t)\sin\left(\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial \mathbf{r}}\cdot\frac{\overrightarrow{\partial}}{\partial \mathbf{k}} - \frac{\overleftarrow{\partial}}{\partial \mathbf{k}}\cdot\frac{\overrightarrow{\partial}}{\partial \mathbf{r}}\right)\right)\rho(\mathbf{r},\mathbf{k},t) \qquad (1)$$

where the Hamiltonian, ω_k, is defined as:

$$\omega_k(\mathbf{r},\mathbf{k},t) = \sqrt{\mathbf{k}^2c^2 + \frac{\omega_p(\mathbf{r},t)^2}{\gamma(\mathbf{r},t)}} \qquad (2)$$

with the fluid equations describing the plasma dynamics in the presence of the photon field; assuming the presence if longitudinal self-consistent electric field:

$$\nabla^2\phi(\mathbf{r},t) = k_{p0}{}^2\left(\frac{n_e(\mathbf{r},t)}{n_0} - 1\right) \qquad (3)$$

$$\gamma(\mathbf{r},t) = \sqrt{1 + \mathbf{p}(\mathbf{r},t)^2 + a(\mathbf{r},t)^2} \qquad (4)$$

$$\frac{\partial \mathbf{p}(\mathbf{r},t)}{\partial t} = m_e c^2 \nabla\phi(\mathbf{r},t) - m_e c^2 \nabla\gamma(\mathbf{r},t) \qquad (5)$$

$$\frac{\partial n_e(\mathbf{r},t)}{\partial t} + \nabla\cdot\left(n_e(\mathbf{r},t)\frac{\mathbf{p}(\mathbf{r},t)}{\gamma(\mathbf{r},t)}\right) = 0 \qquad (6)$$

where ϕ is the scalar potential, n_e the electron density, $\mathbf{p}$ is the electrons momentum. A first order expansion of the stochastic Wigner-Moyal equation, yields a Vlasov like equation for the photons.

Following the standard procedures of the Fourier transform, and making the usual linearizations, we obtain the general nonlinear dispersion relation:

$$1 - \frac{\omega_{p0}{}^2}{\gamma_0\omega_L{}^2} = \frac{a_0{}^2\omega_{p0}{}^2\omega_{k0}(\mathbf{k}_0)}{4\gamma_0{}^3}\left(\frac{\mathbf{k}_L{}^2c^2}{\omega_L{}^2} - \left(1 - \frac{\omega_{p0}{}^2}{\gamma_0\omega_L{}^2}\right)\right)\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_{k0}(\mathbf{k})}\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right) - \rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{\omega_{k0}\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right) - \omega_{k0}\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right) - \omega_L} \qquad (7)$$

where the unperturbed Hamiltonian is given by $\omega_{k0}(\mathbf{k}) = \sqrt{\mathbf{k}^2c^2 + \frac{\omega_{p0}{}^2}{\gamma_0}}$, ω_L is the frequency of the perturbation, $\mathbf{k}_L$ its wave number, γ_0 is defined as $(1 + a_0{}^2)^{1/2}$, and ω_0 is the unperturbed plasma frequency. Note that in the absence of laser, $a_0 = 0$, we recover the dispersion relation for plasma oscillations.

Let us now consider a monoenergetic photon beam, which is equivalent to a plane wave, with a phase space density given by:

$$\rho_0(\mathbf{k}) = n_{0\gamma}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_0) \tag{8}$$

where $n_{0\gamma}$ is the photon density. The integral in the dispersion relation is easily performed in this case:

$$\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\omega_{k0}(\mathbf{k}_0)}{\omega_{k0}(\mathbf{k})}\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)-\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{\omega_{k0}\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)-\omega_{k0}\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)-\omega_L} = \frac{2}{D_+}+\frac{2}{D_-} \tag{9}$$

where we have defined:

$$D_\pm = \pm 2\left[\omega_L \pm \omega_{k0}(\mathbf{k}_0) \mp \omega_{k0}(\mathbf{k}_0 \pm \mathbf{k}_L)\right]\frac{\omega_{k0}\left(\mathbf{k}_0 \pm \frac{\mathbf{k}_L}{2}\right)^2}{\omega_{k0}(\mathbf{k}_0)} \tag{10}$$

3. GROWTH RATES OF UNSTABLE MODES

To study the Raman forward and modulation instability we set $\mathbf{k}_L \,//\, \mathbf{k}_0$. First we consider the underdense scenario ($k_L \ll k_0$) in the weakly relativistic limit ($a_0 \ll 1$), to obtain:

$$\left(\omega_L^{\ 2}-\omega_{p0}^{\ 2}\right)\left[\frac{D_+D_-}{D_++D_-}+\frac{a_0^{\ 2}}{2}\omega_{p0}^{\ 2}\right]+\omega_L^{\ 2}\frac{a_0^{\ 2}}{2}\frac{D_+D_-}{D_++D_-} = \frac{a_0^{\ 2}}{2}\omega_{p0}^{\ 2}c^2k_L^{\ 2} \tag{11}$$

where, in this approximation, we have:

$$D_\pm \approx \left[\omega_L^{\ 2} \pm 2(\omega_0\omega_L - c^2k_0\cdot k) - c^2k_L^{\ 2}\right] \tag{12}$$

We recover the results in [1], with an extra term in the left hand side of the dispersion relation, which takes in account the separate evolution of n_e and γ. The maximum growth rates are not affected, but there is a shift in k_L that maximizes RFS. We thus have for the MI:

$$\Gamma_{\max}(a_0) = \omega_{p0}\left(\frac{\omega_{p0}}{ck_0}\right)^2\frac{a_0}{2} \tag{13}$$

which is the result obtained by Mckinstrie and Bingham [1].

For the case of RFS we have determined the maximum growth rate for all a_0, and we have obtained:

$$\Gamma_{\max}(a_0) = \frac{\omega_{p0}^{\ 2}}{2ck_0}\frac{a_0}{1+a_0^{\ 2}} \tag{14}$$

which is the result obtained by Mori *et al* [2], in the quasi-static limit. Note that in the weakly relativistic limit we reproduce the maximum growth rate given by the previous dispersion relation, recovering again the results in ref. [1].

For the backscattering regime, and for all intensities, we can also obtain the maximum growth rate:

$$\Gamma_{\max}(a_0) = \frac{\sqrt{3}}{2}\left(\frac{a_0^{\;2}}{1+a_0^{\;2}}\right)^{1/3}\omega_0 \tag{15}$$

which reproduces Antonsen and Mora [3], in the weakly relativistic limit.

For the SF regime we set $\mathbf{k}_L \perp \mathbf{k}_0$, and we obtain in the weakly relativistic regime:

$$\Gamma_{\max}(a_0) = \frac{\omega_{p0}}{4}\left(\frac{\omega_{p0}}{ck_0}\right)\frac{a_0^{\;2}}{1+a_0^{\;2}} \approx \omega_{p0}\left(\frac{\omega_{p0}}{ck_0}\right)\frac{a_0^{\;2}}{4} \tag{16}$$

This formula is valid as long as:

$$a_0 << \left(\frac{4\omega_0}{\omega_{p0}}\right)^2 \tag{17}$$

In the limit $a_0 >> 1$, the growth rate is:

$$\Gamma_{\max}(a_0) = \frac{\omega_{p0}}{2ck_0}\sqrt{\omega_{p0}ck_0\left(\frac{1}{a_0}\right)^{3/2} + \frac{\omega_{p0}^{\;2}}{8}\left(\frac{1}{a_0}\right)^2} \tag{18}$$

Setting a small component perpendicular to $\mathbf{k}_0$ and parallel to $\mathbf{k}_L$, and assuming a distribution function in the form:

$$\rho_0(\mathbf{k}) = \frac{n_{0\gamma}}{2}(2\pi)^3\left[\delta(\mathbf{k}-\mathbf{k}_0-\mathbf{k}_{0\perp})+\delta(\mathbf{k}-\mathbf{k}_0+\mathbf{k}_{0\perp})\right] \tag{19}$$

we can obtain a threshold for SF, given by:

$$\frac{a_0^{\;2}}{32}\frac{1}{\gamma_0^{\;3}}\frac{\omega_{p0}^{\;2}}{c^2}W_0^{\;2} > 1 \Rightarrow P(a_0) > 32\frac{c^2}{\omega_{p0}^{\;2}}(1+a_0^{\;2})^{3/2} \tag{20}$$

In the weakly relativistic regime, eq. (21) reproduces the standard threshold for SF.

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