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Fractional diffusion models of transport in magnetically confined plasmas

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I.- Introduction

Motivated by the need to develop models of non-diffusive transport in magnetically confined plasmas, we recently introduced a class of models based on the use of fractional derivative operators. These models incorporate in a unified way non-Fickian transport, non-Markovian ("memory") effects, and non-diffusive scaling. At a microscopic level, the models describe an underlying stochastic process without characteristic spatio-temporal scales that generalizes the Brownian random walk. As a concrete case study, we considered transport of tracers in three-dimensional, pressure-gradient-driven turbulence and showed that there is quantitative agreement between the turbulence transport calculations and the fractional diffusion model [1]. The goal of the present paper is to extend the model discussed in Ref.[1] by incorporating finite-size domain effects, boundary conditions, sources, spatially dependent diffusivities, and general asymmetric fractional operators. These additions are critical in order to order to go beyond tracers transport calculations. In particular, we show that the extended fractional model is able to reproduce within a unified framework some of the phenomenology of non-local, non-diffusive transport processes observed in fusion plasmas, including anomalous confinement time scaling, "up-hill" transport, pinch effects, and on-axis peaking with off-axis fuelling. Recently, similar phenomenology has been described in the context of a probabilistic transport model [2].

II.- Transport model

Our starting point is the fractional model discussed in Ref.[1]

$$ {}_0^cD_t^\beta\, T = l\chi_l\ {}_aD_x^\alpha\, T + r\chi_r\ {}_xD_b^\alpha\, T \qquad (1)$$

where T is the transported scalar (e.g. temperature or density) the left hand side is the fractional Caputo time derivative of order β, and the first and second terms on the right hand side are the left and right Riemann-Liouville (RL) fractional derivative operators of order $m-1 \le \alpha < m$, with m an integer,

$$ {}_aD_x^\alpha\, T = \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_a^x \frac{T(y)}{(x-y)^{\alpha+1-m}}\,dy \qquad {}_xD_b^\alpha\, T = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_x^b \frac{T(y)}{(y-x)^{\alpha+1-m}}\,dy \qquad (2)$$

In the study of tracers discussed in Ref.[1] χ_l, χ_r we assumed constant, and $l = r$. Also, in that calculation only a negligible amount of tracers left the integration domain, and based on this it was assumed that $(a,b) = (-\infty, \infty)$. However, in the study of transport in bounded domains, a and b are finite, and one encounters the nontrivial problem of the singular behaviour of the left and right RL fractional operators at the lower and upper limits respectively [3]. Following what is customary done for the fractional time derivative, we propose here to circumvent this problem by defining the fractional derivatives in space in the Caputo sense. That is, we regularise the singularities by subtracting the boundary terms. Another approximation that needs to be relaxed is the constancy of χ_l, χ_r. To achieve this we write the model is the flux conserving form

$$ {}_0^c D_t^\beta\, T = -\frac{\partial}{\partial x}\left[\, q_\ell + \; q_r + q_G \right] + S(x,t) \qquad (3)$$

where the left and right fractional fluxes are defined as:

$$ q_\ell = -\, l\, \chi_l\, {}_a D_x^{\alpha-1}\left[T(x) - T(a) - T'(a)(x-a)\right] \qquad (4a)$$

$$ q_r = r\, \chi_r\, {}_x D_b^{\alpha-1}\left[T(x) - T(b) + T'(b)(b-x)\right] \qquad (4b)$$

Note that, in addition to the fractional fluxes, we are including a diffusive flux $q_G = -\chi_G\, \partial_x\, T$, and a source term *S(x,t)*. In the following sections we present numerical solutions of the fractional model in Eqs.(3)-(4). We consider $\beta = 1$, $1 < \alpha < 2$, zero flux boundary conditions, $q_L + q_R + q_G = 0$, at $x = 0$, and $T(x = L) = 0$. Consistent with the idea that transport in the core is predominantly diffusive we assume a constant Gaussian diffusivity χ_G, and an anomalous diffusivity profile of the form $\chi_l = \chi_r = (\chi_n/2)\left[1 + \tanh[(x - A)/B]\right]$ (with χ_n, A, B constants) that increases with *x* and vanishes at the core *x=0*.

II. Confinement time scaling

A problem of fundamental importance in fusion research is the scaling of the confinement time $\tau = 1/P\int_0^L T(x)\,dx$, $P = \int_0^L S(x)\,dx$ with the domain size L. According to the standard diffusion paradigm $\tau \sim L^2$. However, L-mode plasmas are known to exhibit anomalous scaling $\tau \sim L^\alpha$ with $\alpha < 2$, Figure 1 shows that the fractional transport model is able to capture this non-diffusive scaling behaviour. In particular, for $\alpha = 1.5$ and $l = r$, a scaling transition is observed from diffusive scaling for small L/L^*, to non-diffusive scaling for large L/L^*, where $L^* = \left(\chi_G/\chi_n\right)^{1/(2-\alpha)}$ and $T^* = \left(\chi_G^\alpha/\chi_n^2\right)^{1/(2-\alpha)}$. A similar result is obtained in the left-asymmetric case $l \neq 0$. $r = 0$. However, consistent with the nonlocal transport properties of the right-fractional derivative, the right-asymmetric case $l = 0 \quad r \neq 0$ shows no anomalous scaling.

III. Uphill-transport and pinch

The fractional model has a built in pinch provided $l \neq r$. In particular, Fig. 2-(a) shows how the spreading of a pulse is accompanied by a shift of the peak of the profile in the case $\alpha = 1.7$. As Fig. 2-(b) shows, this is due to the existence of an "up-hill", non-Fickian transport region in which the flux has the *same* sign as the gradient of the profile. This pinch effect can also be observed in "cold" pulse perturbations of steady profiles in a bounded domain.

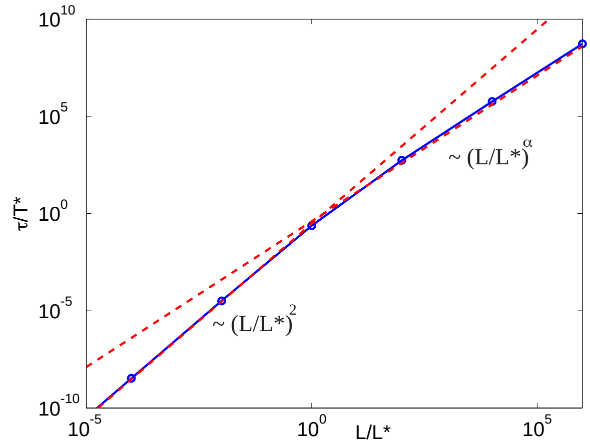

Fig.1 Confinement time scaling

In particular, as Fig.3 shows, the fractional model with $\alpha = 1.25$, and $l \neq 0$ $r = 0$ gives rise to a non-local, fast inward propagation of the pulse. The panel on the left of Fig. 3 shows the perturbed profile at the initial time (in red) and three successive times (in blue). The panel on the right shows a space-time plot of the field T with blue (red) denoting small (large) values.

IV. Off –axis fuelling

As a third example of non-diffusive transport we consider the problem of off-axis fuelling. Figure 4-(a) shows the steady state profile corresponding to a localized, off-centre source S according to the standard diffusion model. The flatness of the profile around the centre is a direct consequence of the locality of the diffusive operator, in particular, as panel (c) shows, the Fickian flux vanishes to the left of the source. On the other hand, as panel (b) shows, the fractional model exhibits on-axis peaking in the right-asymmetric $l = 0$

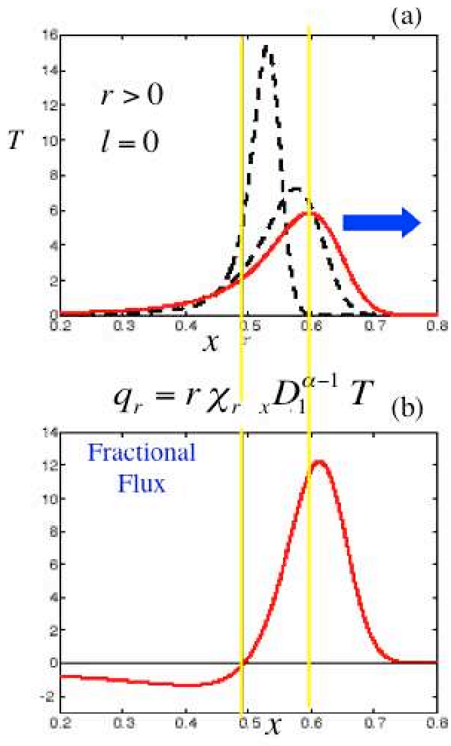

Fig. 2 Pulse decay

$r \neq 0$ case with $\alpha = 1.5$. Panel (d) shows the corresponding fluxes. As expected, the Fickian flux (green) is positive, but the right-fractional flux (shown in blue) is negative and gives rise to the inward transport causing the central peaking. Consistent with the transport properties of the left fractional derivative operator, no on-axis peaking is observed in the left-asymmetric $l \neq 0$ $r = 0$ case.

In summary, we have presented an extension of the fractional model discussed in Ref.[1] that combines regular diffusion with general asymmetric fractional diffusion operators, and incorporates variable diffusivities, finite size domain effects, sources, and physical boundary conditions. Numerical results indicate that the model provides, within a unified framework, a phenomenological description of a variety of non-diffusive processes observed in fusion plasmas.

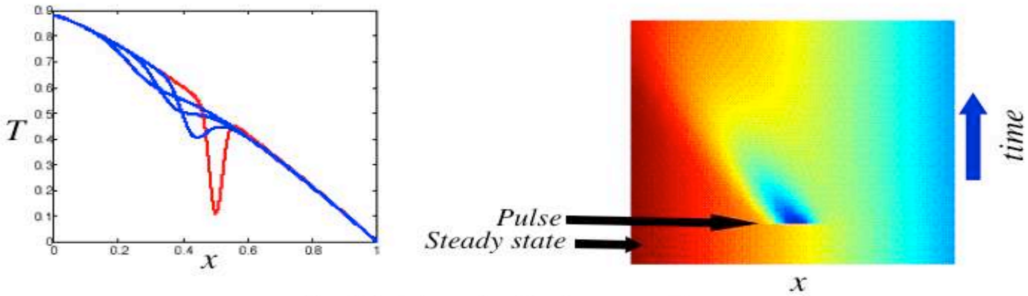

Fig. 3 Fast, inward cold pulse propagation

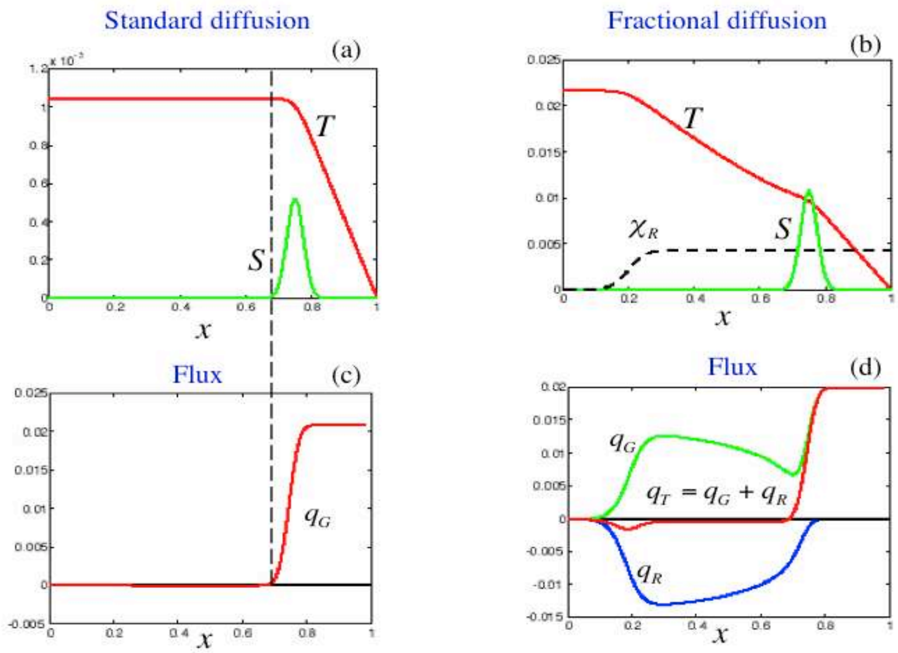

Fig. 4 Off-axis fuelling

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[3] Podlubny, Fractional Differential Equations, (Academic Press, San Diego, 1999).