Non-ideal MHD effects on the nonlinear growth of $m = 1$ internal modes

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We consider here the $m = n = 1$ purely resistive mode [1] that is ideally marginally stable and wish to understand the growth of the $m = 1$ resistive mode up to its nonlinear saturation, on the basis of some striking numerical simulations performed by Aydemir [2] and previous observations [3]. Within the reduced MHD framework in cylindrical coordinates and some given q-profile [2], the time behavior of the kinetic energy in the $m = 1$ mode amounts to an initial exponential growth consistent with the linear regime, followed by a transient stage where the growth rate decreases, that is brutally interrupted by a second exponential growth in the nonlinear regime. This second exponential stage eventually terminates, as the kinetic energy in the $m = 1$ mode saturates which coincides with the completion of magnetic reconnection.

The reduced MHD system under consideration assuming helical symmetry reads

$$\frac{\partial U}{\partial t} = [\phi, U] + [J, \psi], \tag{1}$$

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta(J - J_0). \tag{2}$$

with usual notations [2, 4, 5]. Consider equilibria such that, for some internal radius $r_{s0} < 1$, $q(r_{s0}) = 1$, that is $\psi_0'(r_{s0}) = 0$. Then, due to the Ohm's law (2), plasma volume divides in two region. Far from the $q = 1$ surface (outer domain), plasma behaves ideally whereas, in the vicinity of r_{s0}, resistivity plays a crucial, destabilizing, role. Linear theory [1] uses asymptotic matching analysis to provide $m = 1$ eigenfunctions in the form $A(t) f_L(r) \exp(i\alpha)$ valid in the whole domain. In the outer (ideal) domain, this solution is valid, that is nonlinear effects are negligible, as long as $A \ll 1$ [4]. Injecting the linear solutions $\psi_1(r, \alpha, t) = A(t)\psi_L(r)\exp(i\alpha)$ and $\phi_1(r, \alpha, t) = A(t)\phi_L(r)\exp(i\alpha)$ into (1)-(2) calls for an amplitude expansion [4, 5]. The particularity of the linear radial eigenfunctions $\psi_L(r)$ and $\phi_L(r)$ is that they have strong gradients inside the critical layer. More precisely, their radial derivatives are of the order of the inverse of the critical layer width, i.e. $\mathcal{O}(\eta^{-1/3})$.

The first step will be to determine the end of validity of the linear regime. In the outer domain, this occurs for A of order one [4] but, in the inner domain, the linear solution breaks earlier. This occurs when mode coupling terms such as $[\phi_1, U_1]$ becomes of the same order order as linear terms, that is for $A \gtrsim \eta^{2/3}$. At this point, $m = 0$ and $m = 2$ components begin to be "fed" nonlinearly by mode coupling terms: the $m = 0$ and $m = 2$ modes are nonlinearly driven.

However, these mode coupling terms, quadratic in A, do not affect the $m = 1$ dynamics so that one could say that the $m = 1$ mode is still linear. At this stage, it is easy to check that the dominant equations on the $m = 1$ component are still the linear ones. This means that the radial structure of the solution should remain close to the linear one. Given that, it is possible to include the correction to the linear theory due to the new location of the critical layer. Because of the $m = 1$ perturbation, the critical layer is not expected to remain fixed at r_{s0}. The real helical magnetic field potential inside the critical layer (for $|r - r_{s0}| \lesssim \eta^{1/3}$) is

$$\psi(r, \alpha, t) = \frac{1}{2} \psi''(r_{s0})(r - r_{s0})^2 + A(t) \psi_L(r) \cos \alpha. \tag{3}$$

In writing down the critical layer equations, the instantaneous surface $r_s(\alpha, t)$, defined by $\partial_r \psi[r_s(\alpha, t)] = 0$, is important as the location where dynamical equations turn singular and non-ideal effects come into play. As mode couplings do not affect the second order $m = 1$ dynamics, one can keep the linear $m = 1$ radial structure but introduce the corrections due to the motion of the $\partial_r \psi = 0$ surface. This yields a differential equation [4, 5] for the $m = 1$ amplitude $A(t)$ valid below the onset of cubic nonlinearities on $m = 1$. This is given by

$$\frac{dA}{dt} = \gamma(t) A(t) \tag{4}$$

with

$$\frac{\gamma(t)}{\left\langle \frac{\psi_0''[r_s(\alpha, t)]}{r_s(\alpha, t)} \right\rangle_\alpha} = \frac{\gamma(t = 0)}{\left\langle \frac{\psi_0''[r_s(\alpha, t=0)]}{r_s(\alpha, t=0)} \right\rangle_\alpha} = \frac{\gamma_L}{\frac{\psi_0''(r_{s0})}{r_{s0}}}, \tag{5}$$

where $\langle \cdot \rangle_\alpha \equiv (2\pi)^{-1} \int_0^{2\pi} \cdot d\alpha$ denotes the $m = 0$ average. As shown in Refs. [4, 5], the differential equation (4), with $\gamma(t)$ given by (5), may actually be approximated by the quadratic expression

$$\frac{dA}{dt} = \gamma_L A(t) - C_0 A^2(t), \tag{6}$$

with

$$C_0 = \frac{q'(r_{s0}) + r_{s0} q''(r_{s0}) - 2r_{s0} q'(r_{s0})^2}{\sqrt{2}\pi r_{s0}^2 q'(r_{s0})}. \tag{7}$$

The solution of (6) is

$$A(t) = \frac{A_0 \exp(\gamma_L t)}{1 + A_0 C_0 / \gamma_L [\exp(\gamma_L t) - 1]} \tag{8}$$

that tends to γ_L/C_0 as $t \to \infty$. It is interesting to note that the curve A/A_0 has a universal form depending only on the rescaled time $\gamma_L t$ and on the parameter $A_0 C_0 / \gamma_L$ containing the q-profile properties at r_{s0}. It is also interesting to note that, at time $t_i = \gamma_L^{-1} \ln[\gamma_L/(A_0 C_0) - 1]$, $A(t)$ possesses an inflexion point so that, around that time, the effective behavior of A is approximately algebraic (being linear). All this assumes that C_0 is positive. A negative C_0 would yield a transient explosive faster-than-exponential behavior. However, the validity of (6) is limited because

some cubic nonlinearities should come into play. For S not too large, so that the instantaneous second order location of the critical layer has some large overlap with the initial linear critical layer at r_{s0} around the X-point, it is possible [4, 5] that those cubic terms show up in a spectacular manner. When collecting terms cubic in A, it turns out that in this overlap domain the convective derivative due to the motion of the critical layer dominates the ordinary time derivative and equilibrates mode coupling terms yielding, together with (6), an effective amplitude equation of the form

$$\frac{dA}{dt} - \gamma_L A + C_0 A^2 + \frac{c}{\eta} A^2 \left(\frac{dA}{dt} - \gamma_{NL} A \right) = 0 \tag{9}$$

with $c = \mathcal{O}(1)$ some constant and γ_{NL} the growth rate reached by the $m = 1$ mode at the onset of cubic terms. This takes place for $A \sim \eta^{1/2}$, so that $\gamma_{NL} \sim \gamma_L \eta^{1/2} \left(1 - \frac{C_0}{\gamma_L} \eta^{1/2} \right)$.

Let us finally investigate the **influence of the q-profile** in the onset of the nonlinear regime. The q-profiles $q(r) = q_0 \left\{ 1 + r^{2\lambda} \left[(q_a/q_0)^\lambda - 1 \right] \right\}^{1/\lambda}$, parameterized by λ, can depict different possible experimental situations (Fig. 1). The low-λ case is consistent with a very peaked
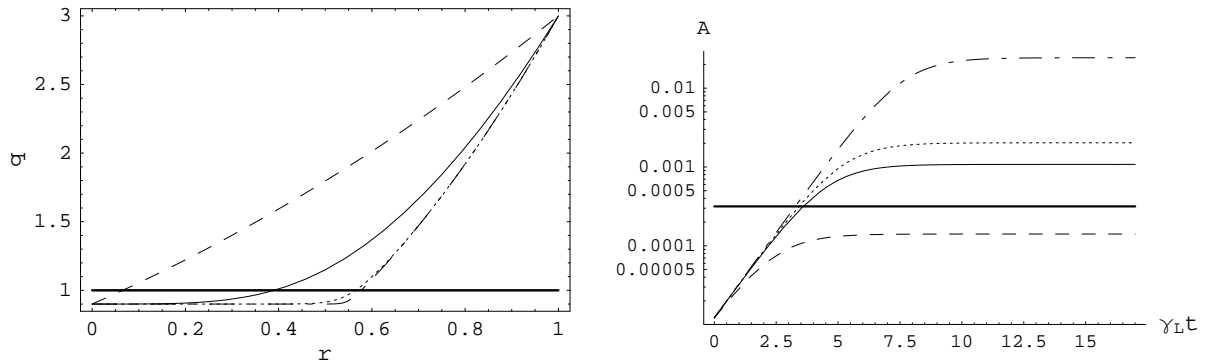

Figure 1: (left) Different q-profiles of the family $q(r) = q_0 \left\{ 1 + r^{2\lambda} \left[(q_a/q_0)^\lambda - 1 \right] \right\}^{1/\lambda}$ with $q_0 = 0.9$ and $q_a = 3$ for $\lambda = 0.6$ (dashed line), $\lambda = 2$ (plain), $\lambda = 10$ (dots) and $\lambda = 30$ (dot-dashed line). (right) Evolution (in linear-log scale) of A as a function of time normalized to linear growth rate for the previous q-profiles (with same plot styles) before the onset of "third order" nonlinear regime. The bold horizontal line marks the threshold of third order convective terms for $A \sim S^{-1/2}$ for the value $S = 10^7$. Initial amplitude is $A_0 = 1.2 \times 10^{-5}$.

current profile. Such a kind of profile was used for instance by Biskamp in his 1991's simulations [6] of the same system. He observed a transition from the linear exponential $m = 1$ growth towards an algebraic behavior. On the contrary, the large λ case coincides with a low shear situation with a flat current profile within the $q = 1$ radius. This is reminiscent of recent experimental investigations undertaken e.g. in JET under the "hybrid" scenario, with a wide area of low magnetic shear and central safety factor close to and below one. Buratti and coworkers

have reported in this case the wide emergence of "slow sawteeth" [7, 8] where the mode has the same spatial structure as the kink-like sawtooth precursor with $n = 1$ but grows very slowly and enters the nonlinear regime at the linear growth rate. Although this case may not strictly correspond to the purely resistive mode, the present analysis on the onset of nonlinear effects should be transposable. Figure 1 may indeed propose an explanation for these observations. In the case of a very peaked q-profile (e.g. with $\lambda = 0.6$), the integration of $m = 1$ amplitude evolution Eq. (8) before the onset of third order convective effects shows that A should saturate before this third order threshold $A \sim \eta^{1/2}$ is reached. This may explain Biskamp's observations [6] of a transition to an algebraic stage with no subsequent nonlinear exponential growth. The $\lambda = 2$ case just corresponds to the q-profile taken by Aydemir in Ref. [2]. Here the third order convective stage showing a second stage of exponential growth can be reached. It is clear from Fig. 1 that the nonlinear growth rate γ_{NL}, that is the growth rate of the $m = 1$ mode when A crosses the threshold ($A \sim \eta^{1/2}$) is smaller than γ_L as A has turned bending. Indeed the simulations of Ref. [2] give the numerical value $\gamma_{NL} \simeq \gamma_L/2$. Yet, for larger values of λ, corresponding to a q-profile of the kind studied by Buratti et al. [7, 8], it is clear from Fig. 1 that γ_{NL} would be almost equal to γ_L: This is an explanation of the fact that, in "slow sawteeth", the $m = 1$ mode enters the nonlinear regime at almost the linear growth rate.

A slightly more developed version of this work is published in Ref. [9].

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