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Sufficient stability condition for axisymmetric equilibrium of flowing plasma

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Stability of static plasma equilibrium can be investigated by use of well-known "energy principle" [1]. The advantage of the energy principle is that there is no necessity of looking for eigenfrequencies and eigenvectors to make a judgement on magnetohydrodynamic (MHD) stability. Another important feature of the principle is that it gives necessary and sufficient condition for spectral (exponential) stability. All of this provides many successful applications of the principle to concrete plasma configurations. However, the attempt to use the same idea to study stability of moving plasma made by Frieman and Rotenberg (FR) [2] fails although the energy principle was formally obtained. Let us briefly describe the idea of the approach.

Consider the linearized equation of motion for plasma displacement $\boldsymbol{\xi}$ in the frame of ideal one-fluid MHD,

$$\rho\ddot{\boldsymbol{\xi}} + 2\rho(\mathbf{V}\cdot\boldsymbol{\nabla})\dot{\boldsymbol{\xi}} - \mathbf{F}(\boldsymbol{\xi}) = 0 \ , \tag{1}$$

where the linearized force operator

$$\mathbf{F}(\boldsymbol{\xi}) = (\boldsymbol{\nabla}\times\delta\mathbf{B})\times\mathbf{B} + (\boldsymbol{\nabla}\times\mathbf{B})\times\delta\mathbf{B} - \delta\rho(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} - \rho(\delta\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} - \rho(\mathbf{V}\cdot\boldsymbol{\nabla})\delta\mathbf{V} - \boldsymbol{\nabla}\delta p - \delta\rho\boldsymbol{\nabla}\Phi$$

is combined of usual perturbed quantities,

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}), \quad \delta\mathbf{V} = (\mathbf{V}\cdot\boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\mathbf{V}, \quad \delta p = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p - \gamma p\,\boldsymbol{\nabla}\cdot\boldsymbol{\xi}, \quad \delta\mathbf{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}).$$

Stationary plasma density, ρ, velocity, $\mathbf{V}$, pressure, p, magnetic field, $\mathbf{B}$, and external potential, Φ, satisfy the equilibrium conditions:

$$\rho(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} + \boldsymbol{\nabla}p = (\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} - \rho\boldsymbol{\nabla}\Phi, \quad \boldsymbol{\nabla}\cdot(\rho\mathbf{V}) = 0 \ ,$$

$$\mathbf{V}\cdot\boldsymbol{\nabla}p + \gamma p\,\boldsymbol{\nabla}\cdot\mathbf{V} = 0, \quad \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) = 0 \ .$$

Dot in Eq. (1) means a partial time-derivative, γ is the adiabatic exponent. Force operator is self-adjoint while the second term in (1) is antisymmetric. Multiplying Eq. (1) by $\dot{\boldsymbol{\xi}}$ and integrating over the space, we find the energy conservation in the form $\dot{E} = 0$, where

$$E(t) = \int \left(\rho\frac{\dot{\boldsymbol{\xi}}^2}{2} - \frac{\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})}{2} \right) d^3r \ . \tag{2}$$

Minimizing E over $\dot{\boldsymbol{\xi}}$, we approach to the FR energy principle,

$$-\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\, d^3r \geq 0 \ . \tag{3}$$

Contrary to the static case ($\mathbf{V} = 0$), where condition (3) appears to be both sufficient and necessary for linear stability, in the case of $\mathbf{V} \neq 0$, condition (3) is known to be normally too strong, and can never be satisfied except for field-aligned flows ($\mathbf{V} \sim \mathbf{B}$) [2] or for those, which may be reduced to the field-aligned flows (see, e.g., [3]).

Energy principle (3) can be improved by use of the Arnold conjecture [4]-[5], following which we have to add to the energy (2) the set of other known integrals of the motion weighted with uncertain Lagrange multipliers.

The most popular integrals of the motion refer to the momentum (or its component) conservation in the presence of symmetry (geometrical or topological). It is just such a symmetry, which allows the system to demonstrate a motional equilibrium, i.e., stationary flows in hydrodynamics. For Eq. (1), this conservation law may be expressed in terms of so called neutral perturbation $\boldsymbol{\xi}_N : \quad \mathbf{F}(\boldsymbol{\xi}_N) = 0, \quad \partial_t \boldsymbol{\xi}_N = 0$. Multiplying Eq. (1) by $\boldsymbol{\xi}_N$ and integrating again over the space, we have $\dot{I} = 0$, where

$$I = \int (\rho\dot{\boldsymbol{\xi}}\cdot\boldsymbol{\xi}_N + 2\rho\boldsymbol{\xi}_N(\mathbf{V}\cdot\boldsymbol{\nabla})\boldsymbol{\xi})\, d^3r \ . \tag{4}$$

For a concrete system geometry, $\boldsymbol{\xi}_N$ can be written explicitly as, e.g., for nested set of magnetic surfaces – see [6, 7]. The improved energy principle, which used integral (4), was derived in [6, 7]. Unfortunately, that stability condition is still not appropriate, because it doesn't solve the above problem of having no sign-definite functional for any stationary plasma flow $\mathbf{V}$.

Variational stability condition allows further improvement when extra invariants are used. E.g., a set of new invariants inherent just in the linearized equation of the motion can be found as follows. Let us differentiate Eq. (1) with respect to time, then multiply it by $\ddot{\boldsymbol{\xi}}$ and integrate over the space. As a result, we find – like in the case of energy but using once again original equation of motion (1) – that the following quantity,

$$E_2 = \frac{1}{2}\int \left\{ \frac{1}{\rho}\left(F(\boldsymbol{\xi}) - 2\rho(\mathbf{V}\cdot\boldsymbol{\nabla})\dot{\boldsymbol{\xi}}\right)^2 - \dot{\boldsymbol{\xi}}\cdot\mathbf{F}(\dot{\boldsymbol{\xi}}) \right\} d^3r \ , \tag{5}$$

is conserved. Invariant (5) is exact for linearized dynamics (1) and cannot be reduced to the conservation of energy (2). In principle, we may continue the procedure and get in the same manner an infinite set of similar invariants. However, to investigate a stability it might be sufficient to involve into our analysis only finite number of the invariants;

this idea was announced in [8]. Here we show that taking into account even one of them, E_2, we are able to improve the stability condition significantly.

Following Arnold's conjecture, we involve into consideration the functional

$$H(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = E_2(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) + \lambda E(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) \ . \tag{6}$$

For more convenience we supplied E in (6) with Lagrange multiplier λ instead of E_2; it is of no importance. There are two ways of how to analyze Eq. (6) looking for a stability. The first one implies the minimization of H and the finding λ, which would provide E equal to its equilibrium value, i.e., to zero. Speaking in other words, variables $\dot{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ in (2) are not absolutely independent but subject to the constraints resulting from the conservation of energy, E [9]. The second one supposes to consider functional (6) for fixed λ and arbitrary $\boldsymbol{\xi}$ and $\dot{\boldsymbol{\xi}}$; this method looks to be easier to proceed. Sign definiteness of the functional guaranties spectral stability in both cases. Here we demonstrate that the use of (6) instead of energy, (2), may result in a reasonable stability condition.

As an example, consider hydrodynamic equilibrium of cold axisymmetric plasma which rotates toroidally. In cylindrical coordinates, $\{r, \varphi, z\}$, $\mathbf{V} = r\Omega\mathbf{e}_\varphi$, where Ω – angular velocity of the rotation. Axial symmetry admits of the separate consideration of each toroidal mode,

$$\boldsymbol{\xi}_m = [\xi_r(r,z)\mathbf{e}_r + \xi_\varphi(r,z)\mathbf{e}_\varphi + \xi_z(r,z)\mathbf{e}_z]e^{im\varphi} \ ,$$

(no coupling). Written in components, Eq. (1) is equivalent to the following scalar equations:

$$\ddot{\xi}_z + 2im\Omega\dot{\xi}_z - m^2\Omega^2\xi_z = 0, \quad \ddot{\xi}_r + 2im\Omega\dot{\xi}_r + (\kappa^2 - m^2\Omega^2)\xi_r = 0 \ .$$

Here κ is an epicyclic frequency: $\kappa^2 = 4\Omega^2 + r(\Omega^2)'_r \geq 0$. Correspondingly, functional (6) can be divided into two independent functionals H_z and H_r, each of them has to be sign-definite. Since $\dot{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ are independent and arbitrary in our analysis, we may require the positive definiteness of the corresponding integrand kernels:

$$\hat{\mathbf{H}}_z = \begin{pmatrix} 3m^2\Omega^2 + \lambda & 2im^3\Omega^3 \\ -2im^3\Omega^3 & m^4\Omega^4 - \lambda m^2\Omega^2 \end{pmatrix} \geq 0 \ , \tag{7}$$

$$\hat{\mathbf{H}}_r = \begin{pmatrix} 3m^2\Omega^2 + \kappa^2 + \lambda & 2im\Omega(m^2\Omega^2 - \kappa^2) \\ -2im\Omega(m^2\Omega^2 - \kappa^2) & (m^2\Omega^2 - \kappa^2)^2 - \lambda(m^2\Omega^2 - \kappa^2) \end{pmatrix} \geq 0 \ . \tag{8}$$

Condition (7) results in the following inequalities:

$$\left\{\begin{array}{rcl} 3m^2\Omega^2+\lambda & \geq & 0\ , \\ m^2\Omega^2(m^2\Omega^2-\lambda) & \geq & 0\ , \\ -m^2\Omega^2(m^2\Omega^2+\lambda)^2 & \geq & 0\ , \end{array}\right.$$

which can always be satisfied by choosing λ:

$$\left[\begin{array}{lcl} \lambda & \geq & 0, \quad \text{if} \quad m=0\ , \\ \lambda & = & -m^2\Omega^2, \quad \text{if} \quad m>0\ . \end{array}\right.$$

Condition (8) needs

$$\left\{\begin{array}{rcl} 3m^2\Omega^2+\kappa^2+\lambda & \geq & 0, \\ (m^2\Omega^2-\kappa^2)(m^2\Omega^2-\kappa^2-\lambda) & \geq & 0, \\ (m^2\Omega^2-\kappa^2)[-\lambda^2-2(m^2\Omega^2+\kappa^2)\lambda-(m^2\Omega^2-\kappa^2)^2] & \geq & 0. \end{array}\right.$$

If $\kappa^2 \leq 0$, the last inequality cannot be satisfied for arbitrary toroidal wavenumber m, therefore, for a stability, our approach requires $\kappa^2 \geq 0$. If it is there are two possible cases:

1. $m^2\Omega^2-\kappa^2 \leq 0 \Rightarrow \lambda \geq -(|\kappa|-m|\Omega|)^2\ .$
2. $m^2\Omega^2-\kappa^2 \geq 0 \Rightarrow -(|\kappa|+m|\Omega|)^2 \leq \lambda \leq -(|\kappa|-m|\Omega|)^2\ .$

It is obvious that in both above cases we can always find λ for any m. Hence, stability condition

$$\kappa^2 \geq 0 \tag{9}$$

is the only restrictive. Note that condition (9) coincides exactly with necessary and sufficient stability condition obtained from local dispersion law (Rayleigh criterion) that confirms the fruitfulness of the suggested approach.

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