Radial diffusion equation for rf-driven current density in tokamaks

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The parallel current driven by radio frequency (rf) waves in a presence of electrostatic fluctuations is theoretically studied in tokamaks. We start with a steady-state relativistic Fokker-Planck equation for an electron distribution function $f(\boldsymbol{r},\boldsymbol{p})$ in a presence of rf waves and low-frequency fluctuations:

$$v_{\|}\boldsymbol{b}\cdot\nabla f - C(f) + \delta\boldsymbol{v}_{\perp}\cdot\nabla f = S_{\mathrm{rf}}, \qquad (1)$$

where $\boldsymbol{b} = \boldsymbol{B}/B$; $v_{\|} = \boldsymbol{v}\cdot\boldsymbol{b}$; C is the Fokker-Planck collision operator; $\delta\boldsymbol{v}_{\perp} = \boldsymbol{B}\times\nabla\delta\phi/B^2$ is the velocity fluctuation due to fluctuating electrostatic potential $\delta\phi$; and S_{rf} is the momentum-space diffusion term due to rf waves. Applying a renormalized perturbation theory to this Fokker-Planck equation, we can obtain a closed set of equations for the ensemble-averaged distribution function and the response function to an infinitesimal external perturbation within the framework of the direct-interaction approximation [1]. These equations for the ensemble-averaged distribution function $g = \bar{f}/f_0$ and the response function G are given by

$$(v_{\|}\boldsymbol{b}\cdot\nabla - \hat{C})g(\boldsymbol{r},\boldsymbol{p}) - \nabla_{\perp}\cdot\int d\boldsymbol{r}'d\boldsymbol{p}'\boldsymbol{F}(\boldsymbol{r}_{\perp}-\boldsymbol{r}'_{\perp})G(\boldsymbol{r}'-\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}')\cdot\nabla_{\perp}g(\boldsymbol{r},\boldsymbol{p}') = \hat{S}_{\mathrm{rf}}, \quad (2)$$

$$-(v_{\|}\boldsymbol{b}\cdot\nabla + \hat{C} + \nabla_{\perp}\cdot\boldsymbol{D}\cdot\nabla_{\perp})G(\boldsymbol{r}-\boldsymbol{r}'',\boldsymbol{p},\boldsymbol{p}'') = \delta(\boldsymbol{r}-\boldsymbol{r}'')\delta(\boldsymbol{p}-\boldsymbol{p}'') \qquad (3)$$

with

$$\boldsymbol{D} = \int d\boldsymbol{r}\,d\boldsymbol{p}'\boldsymbol{F}(\boldsymbol{r}_{\perp})G(\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}'), \qquad (4)$$

where $\hat{S}_{\mathrm{rf}} = S_{\mathrm{rf}}/f_0$ and $\hat{C}(g) = C(gf_0)/f_0$ with the relativistic Maxwellian f_0; $\boldsymbol{F}(\boldsymbol{r}_{\perp}-\boldsymbol{r}'_{\perp}) = \overline{\delta\boldsymbol{v}(\boldsymbol{r}_{\perp})\delta\boldsymbol{v}_{\perp}(\boldsymbol{r}'_{\perp})}$; $\overline{\cdot}$ denotes the ensemble average due to fluctuations; the spatial Markovian approximation is used in (2); and in addition the momentum-space Markovian approximation is used in (3). In this paper, from the rather complicated closed set of equations (2) and (3), we derive a more tractable radial diffusion equation for the rf-driven current density.

Introducing the function χ_1 determined from the equation

$$v_{\|}\boldsymbol{b}\cdot\nabla\chi_1 + \hat{C}(\chi_1) = e\nu_c v_{\|}B, \qquad (5)$$

and using (2), we derive the expression for the rf-driven current density in the form:

$$\langle BJ\rangle = \langle BJ\rangle_0 + \frac{1}{\nu_c}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left\langle\int d\boldsymbol{p}\,f_0(p)\mathcal{A}g(\boldsymbol{r},\boldsymbol{p})\right\rangle, \qquad (6)$$

where $r = |\boldsymbol{r}_\perp|$, $\nu_c = 4\pi\bar{n}e^4\log\Lambda/m^2c^3$, $\langle\,\cdot\,\rangle$ denotes the flux-surface average,

$$\langle BJ\rangle_0 = \frac{1}{\nu_c}\langle\int d\boldsymbol{p}\chi_1 S_{\mathrm{rf}}\rangle \tag{7}$$

is the rf-driven current density in the absence of fluctuations, and

$$\mathcal{A} = \int d\boldsymbol{p}'d\boldsymbol{r}'F(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp)\chi_1(\boldsymbol{p}',\boldsymbol{r}')G(\boldsymbol{r}' - \boldsymbol{r},\boldsymbol{p},\boldsymbol{p}') \tag{8}$$

with $F = \boldsymbol{e}_r\cdot\boldsymbol{F}\cdot\boldsymbol{e}_r$. Furthermore, we define the another function χ_2 satisfying

$$v_\parallel\boldsymbol{b}\cdot\nabla\chi_2 + \hat{C}(\chi_2) = \nu_c\mathcal{A}. \tag{9}$$

When the broadening of the current profile due to fluctuations is small, the mean-squared width of the flux-surface averaged current density driven by rf power absorption at $\boldsymbol{r}_\perp = \boldsymbol{r}_{\perp 0}$ is written as

$$\sigma_G^2 = \frac{\int d\boldsymbol{r}_\perp(\boldsymbol{r}_\perp - \boldsymbol{r}_{\perp 0})^2\langle BJ\rangle}{\int d\boldsymbol{r}_\perp\langle BJ\rangle} \simeq -\frac{\int d\boldsymbol{r}_\perp\langle\int d\boldsymbol{p}\chi_2 S_{\mathrm{rf}}\rangle}{\int d\boldsymbol{r}_\perp\langle\int d\boldsymbol{p}\chi_1 S_{\mathrm{rf}}\rangle}. \tag{10}$$

We now apply the approximate method discussed in ref. 2 to the equation (6). Following this method, we rewrite the equation (6) in the form:

$$\frac{1}{r}\frac{\partial}{\partial r}rD_{\mathrm{rf}}\frac{\partial}{\partial r}\langle BJ\rangle - \nu_c\langle BJ\rangle = -\nu_c\langle BJ\rangle_0, \tag{11}$$

where the diffusion coefficient D_{rf} is written as

$$D_{\mathrm{rf}} \simeq -\frac{\langle\int d\boldsymbol{p}\chi_2 S_{\mathrm{rf}}\rangle}{\langle\int d\boldsymbol{p}\chi_1 S_{\mathrm{rf}}\rangle} = -\left.\frac{\langle\int_{p_\parallel^{(-)}}^{p_\parallel^{(+)}} dp_\parallel S_{\mathrm{w}}\left(\frac{\partial}{\partial\gamma} + N_\parallel\frac{\partial}{\partial p_\parallel}\right)\chi_2\rangle}{\langle\int_{p_\parallel^{(-)}}^{p_\parallel^{(+)}} dp_\parallel S_{\mathrm{w}}\left(\frac{\partial}{\partial\gamma} + N_\parallel\frac{\partial}{\partial p_\parallel}\right)\chi_1\rangle}\right|_{\gamma = N_\parallel p_\parallel + l\omega_c/\omega} \tag{12}$$

with

$$S_{\mathrm{w}} = |U|^2\left(\frac{\partial}{\partial\gamma} + N_\parallel\frac{\partial}{\partial p_\parallel}\right)\bar{f}, \tag{13}$$

where $N_\parallel = k_\parallel c/\omega$; $\gamma = \sqrt{1 + p^2}$; $a = N_\parallel p_\perp\omega/\omega_c$; $U = E_\parallel p_\parallel J_l(a) + [E_-J_{l-1}(a) + E_+J_{l+1}(a)]p_\perp/\sqrt{2}$; l is the harmonic number; ω_c is the electron cyclotron frequency; ω, $k_\parallel$ and $\boldsymbol{E}$ are the frequency, the parallel wave number and the amplitude of injected waves; J_l is the Bessel function; and the momentum is normalized by mc with the electron rest mass m. The upper and lower limits of the integrals in (12) for the cyclotron damping and the Landau damping are given by $p_\parallel^{(\pm)} = \left[N_\parallel l\omega_c/\omega - \sqrt{(l\omega_c/\omega)^2 - 1 + N_\parallel^2}\right]/(1 - N_\parallel^2)$ for $l = 1,2,\cdots$, and $p_\parallel^{(-)} = 1/\sqrt{N_\parallel^2 - 1}$, $p_\parallel^{(+)} = \infty$ for $l = 0$. We note here that the expression (10) for the mean-squared width can be reproduced from the diffusion equation (11).

The wave-induced flux S_{w} includes the unknown function $\bar{f}$. However, the expression for D_{rf} which is the ratio of the linear response to S_{w} is relatively insensitive to

the detail of the wave-induced flux. Therefore, we can estimate the diffusion coefficient using such the approximate (or model) function for $\bar{f}$ as Maxwellian and the distribution function in the absence of fluctuations. In the current-drive concerning the high-energy resonant electrons, the diffusion coefficient $D_{\rm rf}$ is well approximated by

$$D_{\rm rf} = -\frac{\left(\frac{\partial}{\partial\gamma} + N_{\|}\frac{\partial}{\partial p_{\|}}\right)\chi_2}{\left(\frac{\partial}{\partial\gamma} + N_{\|}\frac{\partial}{\partial p_{\|}}\right)\chi_1}\Bigg|_{p_{\|}=p_{\|}^{(-)}, \quad \gamma=N_{\|}p_{\|}^{(-)}+l\omega_c/\omega}, \tag{14}$$

where all the quantiles are now evaluated at the position of resonance.

The function χ_1 is well studied in the neoclassical transport theory and, apart from the normalization, it is considered as the perturbed distribution function in the presence of the parallel d.c. electric field. The function χ_1 for bulk electrons in the banana regime is given by

$$\chi_1 = -ec\mu^{-2}{\rm H}(\lambda_c - \lambda)\Lambda(\lambda)K(x), \tag{15}$$

where $v_e = \sqrt{2T/m}$, $x = v/v_e$, $\mu = mc^2/2T$, H is the Heaviside step function, T is the electron temperature, and $\Lambda(\lambda) = (\langle B^2\rangle/2f_c)\,\sigma\int_\lambda^{\lambda_c}d\lambda/\langle\sqrt{1-\lambda B}\rangle$ with $\sigma = v_{\|}/|v_{\|}|$, $\lambda = (1 - v_{\|}^2/v^2)/B$, $\lambda_c = 1/B_{\rm max}$, and $f_c = (3\langle B^2\rangle/4)\int_0^{\lambda_c}d\lambda\lambda/\langle\sqrt{1-\lambda B}\rangle$. The function $K(x)$ can be obtained using the expansion in a series of the generalized Laguerre polynomials of order $3/2$. We here use the following approximate expression for $K(x)$ obtained by retaining the first two expansion terms:

$$K(x) = \frac{3}{4}\sqrt{\pi}x\left[\alpha'_{1s} - \left(\frac{5}{2} - x^2\right)\alpha'_{2s}\right], \tag{16}$$

where $\alpha'_{1s} = (\mu_{e3}+\sqrt{2}+13Z/4)/D_e$ and $\alpha'_{2s} = (-\mu_{e2}+3Z/2)/D_e$ with $D_e = (\mu_{e3}+\sqrt{2}+13Z/4)(\mu_{e1}+Z)-(\mu_{e2}-3Z/2)^2$, the effective ion charge Z and the conventional viscous coefficients μ_{ek} ($k = 1,2,3$). The function χ_1 for the high energy electrons is obtained using the high-energy form for the collision operator $\hat{C} = \nu_c mc^3[-(1/v^2)\partial/\partial p + (1 + Z)m/(2vp^2)\partial/\partial\zeta(1 - \zeta^2)\partial/\partial\zeta]$, where $\zeta = p_{\|}/p$. This function is given by

$$\chi_1 = -ecB\zeta\left(\frac{\gamma+1}{\gamma-1}\right)^{(Z+1)/2}\int_1^\gamma d\gamma'R(\gamma'), \tag{17}$$

where $R(\gamma) = [(\gamma-1)/(\gamma+1)]^{(Z+1)/2}(\gamma^2-1)/\gamma^2$. We have neglected the toroidal effect for the high energy electrons.

In the similar way, we can calculate the function χ_2 by assuming the explicit form of the correlation function. We now assume the correlation function of fluctuating electrostatic potential in the form:

$$\overline{\delta\phi(\boldsymbol{r}_\perp)\delta\phi(\boldsymbol{0})} = \phi_0^2\exp(-r^2/2\lambda_\perp^2). \tag{18}$$

The analytic expressions for the function χ_2 are obtained in the weak and strong turbulent limits. In the weak turbulent limit, the function χ_2 is derived in the form:

$$\chi_2^{(q)} = ec\nu_c\lambda_\perp^2\xi_c^2$$

$$\times \begin{cases} \mathrm{H}(\lambda_c - \lambda)\Lambda(\lambda)\left(\frac{3}{4}\sqrt{\pi}\right)^3\mu^{-5}x\left[\alpha_{1q} - \left(\frac{5}{2} - x^2\right)\alpha_{2q}\right] & \text{for bulk electrons} \\ \frac{1}{2}B\zeta\left(\frac{\gamma+1}{\gamma-1}\right)^{(Z+1)/2}\int_1^\gamma d\gamma' R(\gamma')[P(p) - P(p')]^2 & \text{for high energy electrons} \end{cases},$$

$$(19)$$

and in the strong turbulent limit, it becomes

$$\chi_2^{(s)} = \frac{1}{\sqrt{2}}ec\nu_c\lambda_\perp^2\xi_c$$

$$\times \begin{cases} \mathrm{H}(\lambda_c - \lambda)\Lambda(\lambda)\mu^{-7/2}\left(\frac{3}{4}\sqrt{\pi}\right)^2 x\left[\alpha_{1s} - \left(\frac{5}{2} - x^2\right)\alpha_{2s}\right] & \text{for bulk electrons} \\ B\zeta\left(\frac{\gamma+1}{\gamma-1}\right)^{(Z+1)/2}\int_1^\gamma d\gamma' R(\gamma')[P(p) - P(p')] & \text{for high energy electrons} \end{cases},$$

$$(20)$$

where $\xi_c = \phi_0/\nu_c\lambda_\perp^2 B$ and $P(p) = p - \tan^{-1}(p)$. The coefficients α_{kq} and α_{ks} for $k = 1, 2$ in (19) and (20) are given by $\alpha_{1q} = [(\mu_{e3} + \sqrt{2} + 13Z/4)(\alpha'_{1s}\hat{\alpha}_1^{(0)} - \alpha'_{2s}\hat{\alpha}_1^{(1)}) - (5/2)(\mu_{e2} - 3Z/2)(\alpha'_{1s}\hat{\alpha}_2^{(0)} - \alpha'_{2s}\hat{\alpha}_2^{(1)})]/D_e$, $\alpha_{2q} = [-(\mu_{e2} - 3Z/2)(\alpha'_{1s}\hat{\alpha}_1^{(0)} - \alpha'_{2s}\hat{\alpha}_1^{(1)}) + (5/2)(\mu_{e1} + Z)(\alpha'_{1s}\hat{\alpha}_2^{(0)} - \alpha'_{2s}\hat{\alpha}_2^{(1)})]/D_e$, $\alpha_{1s} = [(\mu_{e3} + \sqrt{2} + 13Z/4)\alpha'_{1s} - (5/2)(\mu_{e2} - 3Z/2)\alpha'_{2s}]/D_e$, and $\alpha_{2s} = [-(\mu_{e2} - 3Z/2)\alpha'_{1s} + (5/2)(\mu_{e1} + Z)\alpha'_{2s}]/D_e$, where $\hat{\alpha}_1^{(0)} = (\mu_{e3} + \sqrt{2} + 13Z/4)/D_e$, $\hat{\alpha}_2^{(0)} = (-\mu_{e2} + 3Z/2)/D_e$, $\hat{\alpha}_1^{(1)} = (5/2)(\mu_{e2} - 3Z/2)/D_e$, and $\hat{\alpha}_2^{(1)} = -(5/2)(\mu_{e1} + Z)/D_e$.

Interpolating the functions χ_1 and χ_2 for bulk electrons and those for high energy electrons, we calculate the diffusion coefficient from (12). In Fig.1, the diffusion coefficients normalized by $\nu_c\lambda_\perp^2$ are shown as a function of ξ_c for the Landau damping $(l = 0)$.

[2]   M. Taguchi: Phys Plasmas **10**, 37 (2003).

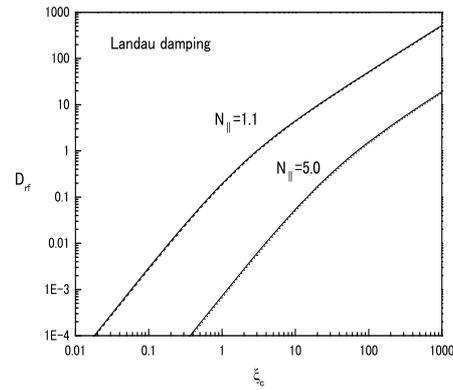[1]   M. Taguchi: Phys. Letters, **A 178**, 165 (1993).

Fig.1: Diffusion coefficient D_{rf} normalized by $\nu_c\lambda_\perp^2$ versus ξ_c. The diffusion coefficients obtained from (14) are shown by dotted curves.