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Connection of interchange instabilities in tokamaks and Parker instabilities in spiral arms of galaxies

J.P. Goedbloed , J.W.S. Blokland , R. Keppens and K.M. Schure

FOM-Institute for Plasma Physics "Rijnhuizen", Association Euratom-FOM
P.O. Box 1207, 3430 BE Nieuwegein, the Netherlands
& Astronomical Institute, Utrecht University, the Netherlands
E-mail: goedbloedrijnh.nl

Abstract

It is well known that interchange instabilities (and associated ballooning modes) in laboratory fusion devices limit the maximum value of the plasma β due to perturbations with a wave vector perpendicular, or nearly perpendicular, to the magnetic field. For those perturbations, the field line bending energy is minimal. The connection between interchange and gravitational (Rayleigh–Taylor) instabilities has frequently been exploited to model the more complicated geometry-dependent dynamics. It is of interest to recall the early work of Newcomb [1] which demonstrated that some of the more subtle gravitational instabilities, the so-called *quasi-interchange instabilities*, require a small parallel component of the wave vector. However, instability is limited to a range of wave vectors that always have a much smaller component parallel than normal to the field lines. Thus, in laboratory plasma theory, it became a kind of paradigm to assume that all major instabilities operate under these conditions.

On the other hand, in astrophysical plasmas like the spiral arms of galaxies, instabilities are operating that produce flow along the magnetic field lines and that require the wave vector to be mainly parallel to the magnetic field: the *Parker instabilities* [2, 3]. Surprisingly, the apparent contradiction of the astrophysical results and the laboratory prejudice appears to have escaped attention. We here present an attempt to reconcile these approaches by fitting both classes of instabilities, the interchange and Parker instabilities, into a wider spectral framework. In the unified theory, a new kind of instabilities appear that we have termed *quasi-Parker instabilities.*

1 Interchanges and quasi-interchanges

In cylindrical and toroidal plasmas, interchange instabilities arise when the negative pressure gradient associated with confinement exceeds the shear of the magnetic field lines (with appropriate factors depending on the magnetic geometry), as expressed by the criteria of Suydam and Mercier. An analogous criterion holds for gravitational interchange instabilities in a plane plasma slab:

$$-\rho N_{\rm B}^2 \equiv \rho' g + \frac{\rho^2 g^2}{\gamma p} \le \tfrac{1}{4} B^2 \varphi'^2 \,, \qquad (1)$$

where N_{B} is the Brunt–Väisäläa frequency, defined by the second expression, and φ' is the magnetic shear, measured by the change of the angle φ of the magnetic field with one of the horizontal axes. Primes indicate derivatives in the vertical direction. In the absence of magnetic shear, stability just depends on the square of the Brunt–Väisäläa frequency: $N_{\mathrm{B}}^2 \geq 0$. [Incidentally, this criterion is identical to the Schwarzschild criterion for convective stability when expressed in terms of the equilibrium temperature gradient.] These criteria are obtained from the marginal equation of motion ($\omega^2 = 0$) in the limit of small parallel wave number ($k_\parallel \to 0$). However, when these two limits are interchanged ($k_\parallel = 0$ and $\omega^2 \to 0$), an entirely different stability criterion is obtained:

$$-\rho N_{\mathrm{m}}^2 \equiv \rho' g + \frac{\rho^2 g^2}{\gamma p + B^2} \leq 0\,, \tag{2}$$

where N_{m} is called the magnetically modified Brunt–Väisäläa frequency. This paradox was resolved by Newcomb [1] who noted that there is a cross-over of two branches of the local dispersion equation with solutions

$$\omega_1^2 = (k_0^2/k_{\mathrm{eff}}^2)\, N_{\mathrm{m}}^2 \qquad \text{(type 1, pure interchanges)}\,, \tag{3}$$

$$\omega_2^2 = \frac{N_{\mathrm{B}}^2}{N_{\mathrm{m}}^2}\,\frac{\gamma p}{\gamma p + B^2}\,\frac{1}{\rho}\,(\mathbf{k}_0 \cdot \mathbf{B})^2 \qquad \text{(type 2, quasi-interchanges)}\,, \tag{4}$$

where the last mode is the first to become unstable when the density gradient is increased. The first expression holds for $k_\parallel = 0$ (where the factor $k_{\mathrm{eff}}^2 \equiv k_0^2 + n^2\pi^2/a^2$ indicates clustering of the modes for vertical mode number $n \to \infty$) and the second is only valid for $k_\parallel \ll k_\perp$, so that field line bending is minimal in both cases.

For a cylinder, analogous expressions were derived by Ware [4], further elaborated by Goedbloed [5] and numerically confirmed by Goedbloed and Hagebeuk [6]:

$$\omega_1^2 = (k_0^2/k_{\mathrm{eff}}^2)\,\frac{2B_\theta^2}{\rho r B^2}\left(p' + \frac{\gamma p}{\gamma p + B^2}\,\frac{2B_\theta^2}{r}\right) \qquad \text{(pure interchanges)}\,, \tag{5}$$

$$\omega_2^2 = \frac{p'}{p' + \dfrac{\gamma p}{\gamma p + B^2}\,\dfrac{2B_\theta^2}{r}}\,\frac{\gamma p}{\gamma p + B^2}\,\frac{1}{\rho}\,(\mathbf{k}_0 \cdot \mathbf{B})^2 \qquad \text{(quasi-interchanges)}\,. \tag{6}$$

Hence, when p' becomes negative (violation of the shearless limit of Suydam's criterion), first the quasi-interchanges become unstable whereas the pure interchanges only become unstable for the much bigger value $-p' = \gamma p\,(\gamma p + B^2)^{-1}\,(2B_\theta^2)/r$, in agreement with the expression derived by Kadomtsev for the z-pinch [7].

2 Parker and quasi-Parker instabilities

So far, the analogy between plasmas with curved magnetic fields and gravitational plasmas is perfect: instability only occurs at the interchange value $k_\parallel = 0$ or close to that value. However, the *Parker instability* in gravitational plasmas, proposed by Parker [2] to explain cloud formation in interstellar gas and the formation of spiral arms in galaxies, operates under precisely the opposite conditions ($k_\perp = 0$). Their growth rate is given by:

$$\omega^2 \approx \left(1 + \frac{\rho N_{\mathrm{B}}^2}{k_{\mathrm{eff}}^2 B^2}\right)\frac{\gamma p}{\gamma p + B^2}\,\frac{1}{\rho}\,(k_0 B)^2\,. \tag{7}$$

In other words, it is very well possible to have instability when the field line bending contributions are not small at all. This is also the case for the magneto-rotational instability. It appears that MHD instabilities occur in astrophysical plasmas under conditions that do not allow instability in laboratory plasmas. Hence, two questions arise: (1) What is so different in those plasmas? (2) How can the two pictures be reconciled?

To address the first question, notice that, in the limit $k_{\rm eff}^2 \to \infty$, the negative contribution of the Brunt–Väisäläa frequency disappears and the modes cluster towards the stable slow magneto-sonic frequency, so that only a finite number of them is unstable. Hence, in contrast to the pure interchanges and quasi-interchanges, the Parker instabilities are essentially *global.*

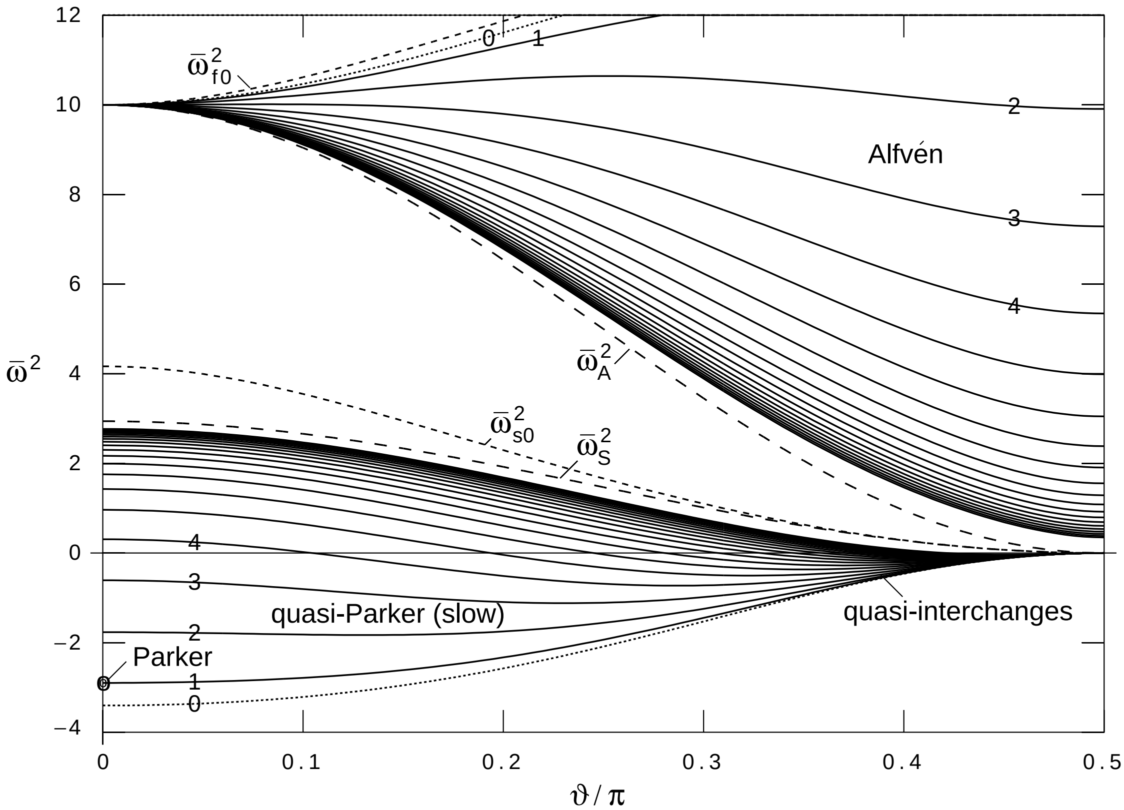

Figure 1: Spectrum of quasi-Parker instabilities; $\alpha = 20$, $\beta = 0.5$, $k_0^2 = 10$, $\gamma = 5/3$.

To answer the second question, we present the results of a calculation of all gravito-MHD waves in a plane gravitating slab, extending the analysis of the textbook by Goedbloed and Poedts [8] based on the wave equation first derived in Ref. [5]. For an exponential atmosphere, $\rho = \rho_0 \exp(-\alpha x)$, $p = p_0 \exp(-\alpha x)$, $B = B_0 \exp(-\frac{1}{2}\alpha x)$, the wave equation can be solved explicitly to yield the three gravitationally modified MHD waves. A dispersion equation is obtained in dimensionless variables:

$$\bar{\omega} = \bar{\omega}(\bar{k}_0, \vartheta, n; \bar{\alpha}, \beta, \gamma)\,, \tag{8}$$

where ϑ is the angle between $\mathbf{k}_0$ and the magnetic field, $\alpha = \rho_0 g/(p_0 + \frac{1}{2}B_0^2)$ is the gravitational parameter and $\beta \equiv 2p_0/B_0^2$. The full set of solutions for arbitrary values of the vertical mode number n yields the spectrum of discrete modes exhibiting Sturmian clustering towards $\omega_{\rm S}^2$ for the slow modes and anti-Sturmian clustering towards $\omega_{\rm A}^2$ for the Alfvén modes. In Figure 1, virtually the complete spectrum is shown (excluding the fast wave solutions that are off-scale) for all directions of $\mathbf{k}_0$ and a particular choice of the parameters α and β. For $\vartheta = 0$ ($k_\perp = 0$) the lowest mode ($n = 1$) is the Parker instability, for $\vartheta \to \pi/2$ ($k_\parallel \to 0$) the complete bundle of unstable modes ($n = 1, 2, \ldots \infty$) represents the quasi-interchanges. The latter modes are a result of the coupling of Alfvén and slow magneto-sonic modes at $k_\parallel \approx 0$ where a slow flow contribution $\parallel \mathbf{B}$ is needed to drive the instability. This contribution becomes large and facilitates the smooth connection of the quasi-interchanges onto the Parker instability through a sequence of slow modes that we have termed *quasi-Parker modes*. As illustrated by Fig. 1, these modes avoid the stabilizing contribution of the Alfvén waves not by making it small (as the interchanges and quasi-interchanges in the region $k_\parallel \approx 0$) but by maximizing the distance to the Alfvénic sub-spectrum (at $k_\perp \approx 0$) so that coupling becomes small for that (entirely different) reason.

The quasi-Parker modes do not have a cylindrical or toroidal counterpart. The reason appears to be that equilibria of astrophysical plasmas represent a much wider class than magnetically confined laboratory plasmas since gravity is independent of the magnetic field so that the narrow confines of $\nabla p = \mathbf{j} \times \mathbf{B}$ are surmounted. This argument also applies to rotation and the resulting magneto-rotational instability in accretion disks.

The slow and Alfvén continuum frequencies $\omega_{\rm A}^2$ and $\omega_{\rm S}^2$ are degenerate here because of the exponential equilibrium chosen. Introducing shear lifts this degeneracy and it modifies the growth rates of the quasi-interchanges significantly, but it does not change the results for the quasi-Parker instabilities very much. In agreement with their global character, these instabilities are robust in the sense of not depending on the subtleties of field line localization that are required to cancel out the stabilizing field line bending contribution of the Alfvén waves in a proper analysis of tokamak stability.

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