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Relativistic quasilinear diffusion in axisymmetric magnetic geometry

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A relativistic bounce-averaged quasilinear diffusion equation is derived to describe stochastic particle transport associated with arbitrary-frequency electromagnetic fluctuations in a nonuniform magnetized plasma. Expressions for the elements of a relativistic quasilinear diffusion tensor are calculated explicitly for magnetically-trapped particle distributions in axisymmetric magnetic geometry in terms of gyro-drift-bounce wave-particle resonances. The resonances can destroy any one of the three invariants of the unperturbed guiding-center Hamiltonian dynamics. While our expressions explicitly involve bounce-averaged wave-particle cyclotron resonances (which appear in the absence of wave-particle phase decorrelation), local wave-particle cyclotron resonances are also uncovered when relevant wave-particle phase decorrelation mechanisms are included within our Hamiltonian formalism.

Recent work by Brizard and Chan [1] presented a full quasilinear theory for nonuniform magnetized plasma in axisymmetric magnetic geometry $\mathbf{B} = \nabla\psi\times\nabla\varphi$ (where the azimuthal angle φ is ignorable) by adopting a high-frequency gyrokinetic Hamiltonian approach. An explicit expression for the 3×3 relativistic quasilinear diffusion tensor was derived in the space of unperturbed guiding-center invariants $\mathbf{I} \equiv (J_g, J_d, E)$ for a magnetically-trapped relativistic particle of mass M and charge q, where J_g is the relativistic gyroaction (canonically conjugate to the ignorable gyroangle ζ), $J_d \equiv q\psi/c$ is the drift action (canonically conjugate to the ignorable azimuthal angle φ), and $E = (\gamma-1)\,Mc^2$ is the unperturbed guiding-center kinetic energy. In the presence of arbitrary-frequency electromagnetic fluctuations $(\varepsilon\delta\phi, \varepsilon\delta\mathbf{A})$, these guiding-center invariants are destroyed ($\dot{I}^i = \varepsilon\,\{I^i,\ \delta H\} \equiv \varepsilon\,\delta\dot{I}^i$) and quasilinear diffusion in $\mathbf{I}$-space occurs.

The high-frequency gyrokinetic Hamiltonian formalism [1] yields a relativistic quasilinear diffusion equation describing resonant transport of relativistic particles that either satisfy the bounce-averaged (denoted by $\langle\cdots\rangle$) wave-particle resonance condition

$$\omega_k \;=\; \ell\,\langle\omega_c\rangle \;+\; m\,\langle\omega_d\rangle \;+\; n\,\omega_b \tag{1}$$

or the local wave-particle resonance condition (at a point s_0 along a magnetic field line)

$$\omega_k \;=\; \ell\,\omega_c(s_0) \,+\, m\,\omega_d(s_0) \,+\, n\,\omega_b, \tag{2}$$

where the fluctuation frequency spectrum is assumed to be discrete $\{\omega_k;\ k = 1,2,...\}$ while $\omega_c \equiv \dot{\zeta}$, $\omega_b = 2\pi/\tau_b \equiv 2\pi(\oint ds/|v_\|\,|)^{-1}$, and $\omega_d \equiv \dot{\varphi}$ denote the relativistic cyclotron, bounce, and azimuthal drift frequencies, respectively. The validity of the bounce-averaged condition (1), which is a universal feature of the action-angle Hamiltonian approach, is based on the assumption that the resonant particles see wave fluctuating fields that are coherent over a bounce period τ_b. Since this assumption is perhaps not realistic for some space-plasma high-frequency (VLF), short-wavelength waves (e.g., whistler-mode waves), wave-particle phase-decorrelation mechanisms must be taken into account to smear out the bounce-averaged resonance condition (1) and leave only the local resonance condition (2). While this problem is often treated by the standard orbit-integration approach [2], we modify our previous work [1] in the present paper by introducing a simple collisional phase-decorrelation mechanism.

We begin with the relativistic high-frequency gyrokinetic equation

$$\frac{dF}{dt} \;\equiv\; \frac{\partial F}{\partial t} \,+\, \{F,\ (E + \varepsilon\,\delta H)\} \;=\; -\nu\,(F - F_0), \tag{3}$$

where $F \equiv F_0 + \varepsilon\,\delta F$ and the perturbed Hamiltonian δH retains full finite-Larmor-radius effects [1], a simple Krook collision term is used to model wave-particle phase decorrelation (ν is a constant collision frequency), and the background guiding-center distribution F_0 satisfies the unperturbed relativistic gyrokinetic equation

$$0 \;=\; \frac{d_0 F_0}{dt} \;\equiv\; \left(\frac{\partial}{\partial t} \,+\, v_\|\,\frac{\partial}{\partial s} \,+\, \omega_c\,\frac{\partial}{\partial \zeta} \,+\, \omega_d\,\frac{\partial}{\partial \varphi}\right) F_0, \tag{4}$$

where $v_\| \equiv \sigma\,|v_\||$ is the parallel guiding-center velocity. In the presence of electromagnetic fluctuations, the relativistic quasilinear diffusion equation for $F_0 \equiv F_0(\mathbf{I}, \tau = \varepsilon^2 t)$ is derived at second order (ε^2) by solving the first-order equation for the perturbed gyrocenter distribution

$$\left(\frac{d_0}{dt} \,+\, \nu\right)\delta F \;=\; -\,\delta\dot{I}^i\,\frac{\partial F_0}{\partial I^i}, \tag{5}$$

whose solution is inserted into the slow-time (second-order) evolution equation

$$\frac{\partial F_0(\mathbf{I},\tau)}{\partial \tau} \;=\; -\,\left\langle \overline{\{\delta F,\ \delta H\}} \right\rangle, \tag{6}$$

where averaging is carried out with respect to the wave time scale and the angles ζ and φ.

The solution to the first-order equation (5) is facilitated by the Fourier expansions $(\delta F, \delta H) \equiv \sum_{m\ell k} (\delta\widetilde{F}, \delta\widetilde{H}) \exp(i\ell\zeta + im\varphi - i\omega_k t)$ and the decompositions [1] for the perturbed gyrocenter distribution

$$\begin{aligned} \delta\widetilde{F} \;&\equiv\; \delta\widetilde{G} + \left(\delta\widetilde{K} - \frac{qv_\parallel}{c}\,\frac{\partial\delta\widetilde{\alpha}}{\partial s} \right) \frac{\partial F_0}{\partial E} \\ &+ \frac{iq}{c} \left[\ell \left(\frac{\partial F_0}{\partial J_g} + \omega_c \frac{\partial F_0}{\partial E} \right) + m \left(\frac{\partial F_0}{\partial J_d} + \omega_d \frac{\partial F_0}{\partial E} \right) \right] \delta\widetilde{\alpha}, \end{aligned} \tag{7}$$

and the perturbed gyrocenter Hamiltonian

$$\delta\widetilde{H} \;\equiv\; \delta\widetilde{K} - \frac{q}{c} \left[v_\parallel \frac{\partial}{\partial s} - i\,(\omega_k - \ell\,\omega_c - m\,\omega_d) \right] \delta\widetilde{\alpha}, \tag{8}$$

where the parallel component of the perturbed vector potential is $\delta\widetilde{A}_\parallel \equiv \partial\delta\widetilde{\alpha}/\partial s$ and the $v_\parallel$-independent perturbed Hamiltonian

$$\delta\widetilde{K} \;\equiv\; \frac{q}{c}(m\omega_d + \ell\omega_c - \omega_k) \int \frac{\delta\widetilde{B}^\psi}{mB}\, ds + \frac{iq}{m}\, \delta\widetilde{E}_\varphi + J_g\omega_c\, \frac{\delta\widetilde{B}_\parallel}{B} \tag{9}$$

is explicitly expressed in terms of covariant or contravariant components of the perturbed electric and magnetic fields. In addition, the nonadiabatic part $\delta\widetilde{G}$ defined in Eq. (7) is the solution to the linear high-frequency gyrokinetic equation

$$e^{i\sigma(\theta+i\delta)}\, \sigma|v_\parallel| \frac{\partial}{\partial s} \left(e^{-i\sigma(\theta+i\delta)}\, \delta\widetilde{G} \right) \;=\; i \left(\omega_k \frac{\partial F_0}{\partial E} + \ell\, \frac{\partial F_0}{\partial J_g} + m\, \frac{\partial F_0}{\partial J_d} \right) \delta\widetilde{K}, \tag{10}$$

where the imaginary phase $\delta(s) \equiv \nu \int_{s_L}^{s} ds'/|v_\parallel|$ models wave-particle phase decorrelation (s_L denotes the lower turning point on the trapped orbit), and the integrated phase is defined as

$$\theta(s) \;\equiv\; \int_{s_L}^{s} \frac{ds'}{|v_\parallel|}\, (\omega_k - \ell\,\omega_c - m\,\omega_d).$$

Next, substituting Eqs. (7)-(8) into Eq. (6) yields the relativistic quasilinear diffusion equation

$$\frac{\partial F_0}{\partial \tau} \;\equiv\; \frac{1}{\tau_b}\, \frac{\partial}{\partial I^i} \left(\tau_b\, D_{\mathrm{QL}}^{ij}\, \frac{\partial F_0}{\partial I^j} \right), \tag{11}$$

where the symmetric 3×3 relativistic quasilinear diffusion tensor is defined as

$$\mathsf{D}_{\mathrm{QL}} \;\equiv\; \sum_{m,\ell,k} \begin{pmatrix} \ell^2 & \ell\omega_k & \ell m \\ \omega_k\ell & \omega_k^2 & \omega_k m \\ m\ell & m\omega_k & m^2 \end{pmatrix} \frac{\tau_b}{2} \left| \left\langle \delta\widetilde{K} \cos\theta \right\rangle \right|^2 \mathrm{Im}\,[\,-\cot(\Theta + i\Delta)\,], \tag{12}$$

where we have neglected collisional effects in the bounce-averaged term $|\langle\delta\widetilde{K}\cos\theta\rangle|^2$ and the bounce-averaged phases are $\Theta \equiv \frac{1}{2}\tau_b\,(\omega_k - \ell\,\langle\omega_c\rangle - m\,\langle\omega_d\rangle)$ and $\Delta \equiv \frac{1}{2}\tau_b\nu$. First, in the absence of collisional effects ($\Delta \equiv 0$), which corresponds to our previous work [1], the last term in Eq. (12) is expressed (making use of the Plemelj formula) as

$$\mathrm{Im}\,(-\,\cot\Theta\,) \;\equiv\; \sum_{n=-\infty}^{\infty}\pi\,\delta(\Theta - n\pi) \;=\; \sum_{n=-\infty}^{\infty}\omega_b\,\delta\left(\omega_k \,-\, \ell\,\langle\omega_c\rangle \,-\, m\,\langle\omega_d\rangle \,-\, n\,\omega_b\right),$$

and, thus, the quasilinear diffusion described by Eq. (11) explicitly involves resonant particles that satisfy the bounce-averaged wave-particle resonance condition (1). The high-frequency gyrokinetic Hamiltonian approach also implicitly contains the local wave-particle resonance condition (2) in the bounce-average integral $\langle\delta\widetilde{K}\,\cos\theta\rangle$, which can be extracted by stationary-phase methods as follows. First, we introduce the bounce-angle Fourier expansion $\delta\widetilde{K}(s) \equiv \sum_n\,\delta\overline{K}_n\,\exp[in\xi(s)]$, where $\xi(s)$ denotes the bounce angle, and the net exponential phase $\phi(s)$ in the bounce-average integral is now

$$\sigma\,\phi(s) \;\equiv\; \sigma\,\theta(s) \;-\; n\,\xi(s) \;=\; \sigma\int_{s_L}^{s}\frac{ds'}{|v_\parallel|}\left(\omega_k - \ell\,\omega_c(s') - m\,\omega_d(s') - n\,\omega_b\right).$$

Next, by using stationary-phase methods, the dominant contribution comes from points s_0 along a magnetic-field line where $0 = \phi'(s_0) = |v_\parallel(s_0)|^{-1}\,[\omega_k - \ell\,\omega_c(s_0) - m\,\omega_d(s_0) - n\,\omega_b]$, which yields the local wave-particle resonance condition (2) provided $v_\parallel(s_0) \neq 0$ (i.e., the local resonance does not occur at the turning points). Lastly, within our simple collisional phase-decorrelation mechanism (where $\nu \ll \omega_c$), the last term in Eq. (12) yields

$$\mathrm{Im}\,[\,-\,\cot(\Theta + i\,\Delta)\,] \;\equiv\; \sum_{n=-\infty}^{\infty}\frac{\Delta}{(\Theta - n\pi)^2 \,+\, \Delta^2},$$

which smears out the bounce-averaged resonance condition $\Theta = n\pi$ (if $\Delta \simeq 1$) and leaves the local wave-particle resonance (2) in the bounce-average integral $\langle\delta\widetilde{K}\,\cos\theta\rangle$ intact.

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