

Four motional invariants in adiabatic equilibria

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Recently published derivations of four stationary motional invariants [1-3] in adiabatic equilibria are presented. The four invariants $(\varepsilon, \mu, I_r, I_{\parallel})$ contains a radial drift invariant I_r , which determines the density radial profile and the diamagnetic drift, and an additional parallel invariant I_{\parallel} that determines the plasma current along the magnetic field. Thus, there are in general more than three stationary invariants for the adiabatic motion of a gyrating particle. The result is valid to first order in the gyro radius, and is applicable to geometries with adiabatic fields, including toroidal as well as open mirror geometry. In axisymmetric tori, the toroidal invariant can replace the longitudinal invariant in the analysis and the radial invariant can be determined from the projected gyro center motion. The four invariants is determined for passing as well as trapped particles. For equilibria with sufficiently small banana widths and radial drift excursions, the radial invariant can to lowest order be approximated by the gyro center value $I_r \approx \bar{r}_0(\mathbf{x}, \mathbf{v})$ of the radial Clebsch coordinate. To this lowest order, the gyro centers drift on a magnetic flux surface.

Invariants are useful in a variety of plasma studies, and it is not possible to foresee in detail all areas where an application of a fourth invariant could be important. It is well known that MHD and kinetic stability, transport and heating are profile sensitive, and a complete set of invariants could be required to develop reliable models with realistic profiles.

The existence of a radial invariant has a range of implications. The dependence on the radial invariant determines the radial profiles of the density and temperature and this gives a diamagnetic drift in a direction perpendicular to the magnetic field. In tokamaks, where the standard set $(\varepsilon, \mu, p_{\varphi})$ of three invariants is insufficient to model a poloidal current, the poloidal current can directly be determined by using the radial invariant in the distribution function. It is also possible to establish a bridge between Vlasov equilibria and ideal MHD with the use of the radial invariant.

To determine the dependence of $I_r(\mathbf{x}, \mathbf{v})$ on the phase space coordinates of (\mathbf{x}, \mathbf{v}) , it is necessary to carry out an adiabatic expansion of the equations of motions. First, a transformation $(x, y, z) \rightarrow (r_0, \theta_0, s)$ to flux coordinates is introduced by, see [1] and [2],

$$\frac{\mathbf{B}}{B_0} = \nabla r_0 \times r_0 \nabla \theta_0$$

$$\nabla s = \hat{\mathbf{B}} + \kappa \nabla r_0 + \eta r_0 \nabla \theta_0$$

where B_0 is a constant, r_0 and θ_0 are the radial and angle Clebsch coordinates, s is the arc length along the magnetic field lines and $\kappa(\mathbf{x})$ and $\eta(\mathbf{x})$ are “geometric” functions associated with the magnetic field geometry. The particle velocity $\mathbf{v} = v_{\parallel} \hat{\mathbf{B}} + \mathbf{v}_{\perp}$ is

$$\mathbf{v}_{\perp} = \frac{(\dot{r}_0 r_0 \nabla \theta_0 - r_0 \dot{\theta}_0 \nabla r_0) \times \hat{\mathbf{B}}}{B/B_0}$$

$$v_{\parallel} = \dot{s} - (\kappa \dot{r}_0 + \eta r_0 \dot{\theta}_0) = \dot{s} - \mathbf{v}_{\perp} \cdot \nabla s \equiv \frac{ds}{d\tau_{\parallel}}$$

and dots stand for time derivatives. The last formula shows that v_{\parallel} is the rate of change of the arc length projected on the flux lines, and the parallel velocity is zero at locations where the motion does not generate a change of the arc length coordinate of the particle. The first order difference between v_{\parallel} and \dot{s} must be included to arrive at exact energy conservation for the gyro center motion in a stationary field.

The motion is split into a rapidly gyrating part and a slowly varying gyro center motion. Bars denote gyro center quantities, and the radial position of the particle is of the form

$$r_0 = \bar{r}_0(\mathbf{x}, \mathbf{v}) + r_{0,\text{gyro}} \quad (1)$$

where $r_{0,\text{gyro}}$ is a “gyro ripple” associated with the gyrations, see [1] and [2], and this gyro ripple is responsible for the diamagnetic current. In a stationary field, the velocity of the guiding center is to first order in the gyro radius determined by the four equations

$$\varepsilon = U_{gc}(\bar{\mathbf{x}}) + \frac{m\bar{v}_{\parallel}^2}{2} = \text{constant} \quad (2a)$$

$$\frac{d\bar{s}}{dt} = \bar{v}_{\parallel} + \bar{\kappa} \frac{d\bar{r}_0}{dt} + \bar{\eta} \bar{r}_0 \frac{d\bar{\theta}_0}{dt} \equiv \bar{v}_{\parallel} + \bar{\mathbf{v}}_{\perp} \cdot \nabla \bar{s} \quad (2b)$$

$$m\Omega_0 \frac{d\bar{r}_0}{dt} = -\frac{1}{\bar{r}_0} \frac{\partial U_{gc}}{\partial \bar{\theta}_0} - \bar{\eta} \frac{\partial U_{gc}}{\partial \bar{s}} + 2(\varepsilon - U_{gc}) \frac{\partial \bar{\eta}}{\partial \bar{s}} \quad (2c)$$

$$m\Omega_0\bar{r}_0 \frac{d\bar{\theta}_0}{dt} = \frac{\partial U_{gc}}{\partial \bar{r}_0} + \bar{\kappa} \frac{\partial U_{gc}}{\partial \bar{s}} - 2(\varepsilon - U_{gc}) \frac{\partial \bar{\kappa}}{\partial \bar{s}} \quad (2d)$$

where $U_{gc}(\bar{\mathbf{x}}) = q\phi + \mu B$ and $\bar{\mathbf{v}}_{\perp}(\bar{\mathbf{x}})$ is the guiding center perpendicular drift,

$$\bar{\mathbf{v}}_{\perp}(\bar{\mathbf{x}}) = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mu}{q} \frac{\mathbf{B} \times \nabla B}{B^2} + \frac{m\bar{v}_{\parallel}^2}{qB^4} \mathbf{B} \times [(\mathbf{B} \cdot \nabla)\mathbf{B}]$$

For a periodic guiding center motion, a longitudinal invariant, which is an exact (not only adiabatic) constant to first order in the gyro radius, can be defined as an integral along the gyro center orbit, i.e.

$$J_{\parallel} = \sqrt{2m} \oint \sqrt{\varepsilon - U_{gc}(S)} dS, \quad (3)$$

where S is a curve parameter for the guiding center orbit [1]. This definition differs somewhat from the parallel action integral introduced by Northrop [4,5] (which is integrated along a magnetic field line), see also [6]. For confined particles, the drift orbit average $I_r = \langle \bar{r}_0 \rangle$ must be constant, and $(\varepsilon, \mu, I_r, J_{\parallel})$ is a set of four independent invariants for the first order adiabatic motion.

In axisymmetric tori fields, the symmetry gives a toroidal invariant p_{φ} , which can replace the parallel invariant in the analysis, and the drift orbit can be projected on a closed curve in the (r, z) plane. The projected gyro center motion determines the radial drift invariant. These properties are seen by introducing the poloidal angle ζ_0 ,

$$\zeta_0(r_0, \theta_0, s) = -\frac{\pi}{2} + \theta_0 \frac{B_0}{\tilde{B}_t(r_0)} + \frac{s}{r_0} \frac{\tilde{B}_p(r_0)}{\tilde{B}(r_0)} = \zeta(r, z) \quad (4)$$

where $\tilde{B}^2 = \tilde{B}_p^2 + \tilde{B}_t^2$ and $\zeta(r, z)$ is specified in [1] and [2]. The relation $\zeta_0(r_0, \theta_0, s) = \zeta(r, z)$ is associated with the linear dependence of $\theta_0(r, \varphi, z)$ and $s(r, \varphi, z)$ on the toroidal angle and the toroidal symmetry of the fields, which implies $U_{gc} = U_{gc}(r_0, \zeta)$, see [1] and [2]. The projected guiding center motion is determined by

$$\frac{d\bar{r}_0}{dt} = -\frac{\tilde{B}_t}{qB\tilde{B}} \frac{1}{\bar{r}_0} \left[\frac{\partial U_{gc}}{\partial \zeta} + 2(\varepsilon - U_{gc}) \frac{1}{B} \frac{\partial B}{\partial \zeta} \right] \quad (5a)$$

$$\frac{d\bar{\zeta}}{dt} = \frac{\tilde{B}_p}{\tilde{B}} \frac{v_{\parallel}}{\bar{r}_0} + \frac{\tilde{B}_t}{qB\tilde{B}} \frac{1}{\bar{r}_0} \left[\frac{\partial U_{gc}}{\partial \bar{r}_0} - 2(\varepsilon - U_{gc}) \frac{c_0}{\bar{r}_0} \right] \quad (5b)$$

where $v_{\parallel} = \sigma \sqrt{2/m} \sqrt{\varepsilon - U_{gc}}$, $\sigma = \pm 1$ determines the direction of the parallel velocity and $c_0(\bar{r}_0, \bar{\zeta})$ is defined in [2]. In straight systems with rotational symmetry, \bar{r}_0 is a motional

invariant and such equilibria are omnigenous with constant values of $c_0 = \tilde{B}_p^2 / B^2$ and $d\bar{\zeta} / dt$.

In axisymmetric tori, passing particles far from a trapped state have a nearly constant $d\bar{\zeta} / dt$, while $\bar{\zeta}$ oscillates for a trapped particle, providing a finite banana orbit width. For each constant values of ε and μ , the solution of Eqs. (5a,b), for passing as well as trapped particles, is a closed curve $\bar{r}_0(\bar{\zeta}) \equiv \bar{r}_0(\sigma, \varepsilon, \mu, \bar{\zeta})$ in the (r, z) plane, where $\bar{\zeta}$ is the curve parameter. For trapped particles, two orbit portions with opposite signs of v_{\parallel} connect at the points where σ changes sign.

The system of guiding center equations provides a radial invariant $I_r = \bar{r}_0(\mathbf{x}, \mathbf{v}) + I_r^{(1)}$ and a toroidal invariant \bar{p}_{φ} , which has the same value as the toroidal invariant of the particle [7].

The general solution of the stationary Vlasov equation is a function of four (not only three, as often stated) invariants. In axisymmetric equilibria, a nearly local Maxwellian distribution function, expressed in terms of the invariants and thereby as a solution of the Vlasov equation, can be written

$$F(\varepsilon, \mu, p_{\varphi}, I_r) = n_0(I_r) \left[\frac{2/m}{\pi k_B T_0(I_r)} \right]^{3/2} e^{-\varepsilon/k_B T_0(I_r)} + F^{(1)}$$

where the correction $F^{(1)}$, and its contribution to the toroidal current, has to be determined from detailed considerations of the transport and heating. With inclusion of the first order finite radial drift excursions and neglecting the contribution from $F^{(1)}$ to the radial force balance, we obtain $\mathbf{j}_{\perp} \times \mathbf{B} \approx \nabla P(r_0)$, which provides a bridge between Vlasov equilibria and ideal MHD, see [2].

References

1. O. Ågren, V. Moiseenko, C. Johansson and N. Savenko, Phys. Plasmas **12**, 122503 (2005).
2. O. Ågren and V. Moiseenko, Phys Plasmas **13**, 052501 (2006).
3. O. Ågren, V. Moiseenko and N. Savenko, Phys Rev E **72**, 026408 (2005).
4. T. Northrop and E. Teller, Phys. Rev. **117**, 215 (1960).
5. T. Northrop, *The adiabatic motion of charged particles*, Vol 21 of Interscience Tracts on Physics and Astronomy, Library of Congress Catalog Card number 63-22462 (Wiley, New York, 1963).
6. M. Kruskal, J. Math. Phys. **4** 806 (1962).
7. R. G. Littlejohn, J. Phys. Fluids, **29**, 111 (1983).