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A three-dimensional dynamical system model for profile resilience and anomalous cross-field transport

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In plasma physics the notion of anomalous transport is used indiscriminately to characterize phenomena giving rise to transport coefficients which exceed the collisional diffusive transport by orders of magnitude. Such transport is caused by the excitation of collective motions resulting from gradient driven instabilities. Similar processes arise in geophysical fluids, and explain rapid transport which cannot be explained by diffusive mechanisms alone. The archetype of a geophysical anomalous transport mechanism is the Rayleigh-Bénard (R-B) convection, driven by a vertical temperature gradient in a fluid horizontally straitified by gravity. In 1963 E. Lorenz developed a dynamical system to describe the nonlinear evolution of this instability, and found deterministic, aperiodic flows (chaos).

The classical description of the R-B instability treats non-convective boundaries. This formulation, however, cannot give transport through the layer which greatly exceeds the heat flux due to thermal diffusion driven by the imposed temperature gradient. Hence this formulation is not suitable to describe anomalous transport.

The R-B instability has its counterpart in flute interchange instabilities in magnetized plasmas, driven unstable by pressure gradients and magnetic field curvature. We derive the R-B model and the Lorenz model for electrostatic interchange instability, and discuss similarities and differences in the underlying physics.

The weakness of the classical formulation of the convection problem becomes apparent in the limit of vanishing diffusivity, and is studied more naturally in a formulation where the diffusivity appears explicitly as coefficients in the equations. In this limit the energy flux through the layer vanishes unless the driving gradient goes to infinity. This unphysical feature is a strong motivation to seek an alternative formulation of the convection problem which allows for finite fluxes, even in the diffusionless limit. We name this formulation the *convective filament* model, as opposed to the traditional *convective cell model*.

For electrostatic flute modes driven unstable by the electron density gradient in a

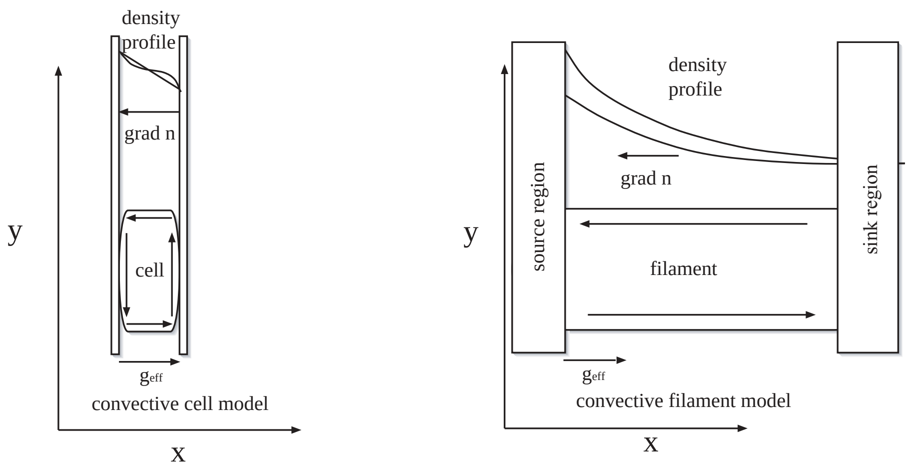

Figure 1: Left: Convective cell model with fixed, non-convective boundaries and periodic profile perturbation. Right: Convective filament model with open convection through boundaries and exponential profile only perturbed by change in scale-length.

plasma confined by a magnetic field with radius of curvature R_0, we can derive equations equivalent to the R-B model

$$(\partial_t + \hat{\mathbf{z}} \times \nabla\varphi \cdot \nabla)\nabla_\perp^2 \psi + \rho\, \sigma\, \partial_y N = \sigma\, \nabla_\perp^4 \psi\,,$$

$$(\partial_t + \hat{\mathbf{z}} \times \nabla\varphi \cdot \nabla) N + \partial_y \psi = \nabla_\perp^2 N\,,$$

where ψ is the stream function, $N = \ln n$ is the logarithm of the perturbed electron density, σ is the Prandtl number and ρ is the analog of the Rayleigh number in a fluid;

$$\sigma = \frac{\eta}{D_{cl}}\,, \quad \rho = \frac{(\Delta N - 2d/R_0) g_{eff} d^3}{\eta D_{cl}}\,.$$

Here η is the ion viscosity due to ion-ion collisions, D_{cl} is the classical cross-field diffusion coefficient due toe elctron-ion collisions, ΔN is the difference in background density logarithm N between the two boundaries separated by a distance d, and g_{eff} is the effective "gravitational" acceleration due to the magnetic field curvature. Lorenz' analysis considers convection rolls on the form

$$\psi = A(t)\sin(Ky)\sin(\pi x)\,, \quad N = B(t)\cos(Ky)\sin(\pi x) - C(t)\sin(2\pi x)\,,$$

which inserted into the R-B model and a change of variables

$$X = \frac{\pi K}{(\pi^2 + K^2)\sqrt{2}} A\,,\ Y = \frac{\pi r}{\sqrt{2}} B\,,\ Z = \pi r\, C\,,\ \ r = \frac{K^2}{(\pi^2 + K^2)^3}\rho\,,\ b = \frac{4\pi^2}{\pi^2 + K^2}\,,$$

yields the standard form of the Lorenz equations

$$\dot{X} = \sigma(Y - X)\,,\ \ \dot{Y} = rX - XZ - Y\,,\ \ \dot{Z} = XY - bZ\,. \qquad \text{(LE-1)}$$

From numerical solutions of the Lorenz equation we can show that the maximal flux carried by these solutions is $\Gamma = (\Delta N D_{cl}/d)\,(3 - 1/r)$. The weakness of this convective cell formulation is apparent in the limit of vanishing diffusivity, $D_{cl} \to 0$, since a finite flux then requires an infinite density gradient. The alternative convective filament formulation assumes an exponential density profile (linear N-profile) with time varying inverse scale length $\Delta N(t) = d/L_n$, and filamentary flute perturbations;

$$\psi(y,t) = a(t)\sin(ky)\,,\ \ N(y,t) = b(t)\cos(ky)\,,\ \ N(x,t) = c(t) + 2d/R_0\,.$$

By introducing the viscous time scale and a change of variables and parameters

$$t \to \eta k^2 t\,,\ U = \frac{1}{kd\eta} a\,,\ V = \frac{g_{eff}}{k^4 d\eta^2} b\,,\ W = \frac{g_{eff}}{k^4 d\eta^2} c\,,$$

$$\chi = \frac{1}{\sigma} = \frac{D_{cl}}{\eta}\,,\ \xi = \frac{2}{k^2 d^2}\chi\,,\ R = \frac{2}{K^6 d^3 \eta^3} g_{eff}\zeta\left(D_{eff} - D_{cl}\right)\,,$$

the R-B model reduces to the dynamical system;

$$\dot{U} = -U - V\,,\ \ \dot{V} = -\chi V - WU\,,\ \ \dot{W} = -\xi W + UV + R\,, \qquad \text{(LE-2)}$$

Here $D_{eff} = 2\Gamma_s/(R_0 n_d)$ is the effective anomalous diffusion coefficient which would be required to drive a flux Γ_s imposed by a plasma source when the density gradient is close to instability threshold. The interesting case is strong flux, i.e. when $D_{eff} \gg D_{cl}$.

It is easily shown the the Lorenz equations (LE-1) can be transformed into (LE-2) by a simple change of variables, hence this is only an alternative formulation of the Lorenz equations. Eqs. (LE-2), however, is more suitable for going to the diffusionless limit $D_{cl} \to 0$, which implies letting $\chi, \xi \to 0$, keeping R finite. Many plasma applications correspond to $\chi, \xi \ll 0$ and $R > 1$, so a study of the ensuing diffusionless Lorenz equations, depending only on one free parameter R, may be physically relevant.

The convective filament model allows arbitrary strong fluxes D_{eff}, and moreover the density profile will make small fluctuations around the threshold profile if $D_{eff} \ll \eta$. These oscillations may be chaotic for $R \lesssim 5$, but will be periodic for $R > 5$, according to the results displayed in Figs. 2 and 3 below.

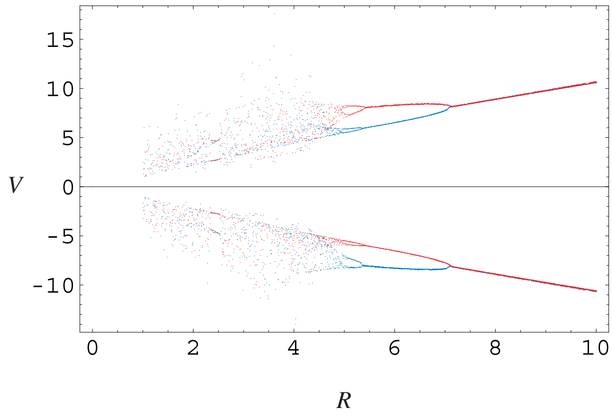

Figure 2: Bifurcation diagram created from Poincaré plot of intersection of trajectory with U-V plane for varying R. Bifurcations where both branches are red are period doblings, while branches with different colors display splitting into different limit cycles.

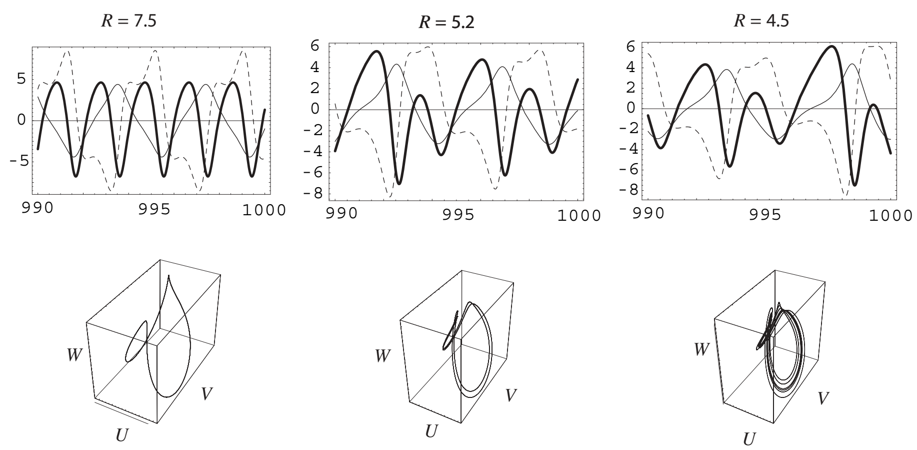

Figure 3: Upper:Time-evolution for $U(t), V(t), W(t)$ (thin full, thin dotted, and thick full curves, respectively), for three values of R. Lower: The corresponding phase-space orbits.