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Variational approach to the study of axial symmetric MHD equilibrium flows

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The object of this report is to derive the variational principle that governs the stationary ideal axisymmetric MHD flow and to show its application to the solution of open-boundary problems. We shall start deriving the governing differential equation and the Bernoulli's constraint, both characterizing the solution. The equations to be considered are:

$$\begin{aligned} \nabla\cdot(\rho\mathbf{v}) &= 0, \qquad \mathbf{v}\cdot\nabla S = 0, && (1)\\ \rho(\mathbf{v}\cdot\nabla)\mathbf{v} &= -\nabla p + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}, && (2)\\ \nabla\times(\mathbf{v}\times\mathbf{B}) &= 0, \qquad \nabla\cdot\mathbf{B} = 0, && (3) \end{aligned}$$

where ρ is the mass density, $\mathbf{v}$ is the plasma velocity, p is the scalar pressure, $\mathbf{B}$ is the magnetic field and S is the entropy per unit mass. In what follows we use cylindrical (r,ϕ,z) coordinates, where ϕ represents the azimuthal angle. The axial symmetry allow us to express the poloidal magnetic field in terms of the derivatives of a flux function ψ as $\mathbf{B}_p = \nabla\psi\times\nabla\phi$, and the perfect conductivity equation yields $\mathbf{v}_p = \kappa(r,z)\mathbf{B}_p$. It is well known that this set of equations presents five field-line constants that can be expressed as functions of ψ :

$$\begin{aligned} 4\pi\rho\kappa &= F(\psi), \qquad S(\rho,p) = S(\psi), && (4)\\ \frac{1}{r}(v_\phi - \kappa B_\phi) &= G(\psi), \qquad rB_\phi - rFv_\phi = H(\psi). && (5) \end{aligned}$$

The Euler equation (2) on the poloidal plane gives the two fundamental equations of the problem: from the component along B_p we obtain the generalized Bernoulli constant:

$$\int (dp/\rho)\bigg|_{\psi=const} + v^2/2 - rv_\phi G = J(\psi), \qquad (6)$$

while the component parallel to $\nabla\psi$ gives the generalized Grad-Shafranov equation

$$\begin{aligned} \left(1-\frac{F^2}{4\pi\rho}\right)\triangle^*\psi \;&-\; F\nabla\left(\frac{F}{4\pi\rho}\right)\cdot\nabla\psi = && (7)\\ &-\; 4\pi\rho r^2\left(J' + rv_\phi G' - \frac{p}{\rho}\frac{S'}{k_B}\right) - (H + rv_\phi F)(H' + rv_\phi F'), \end{aligned}$$

where $\triangle^*\psi$ is defined as $\triangle^*\psi = r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}) + \frac{\partial}{\partial z}\frac{\partial\psi}{\partial z}$.

Since the model is non dissipative we search for an equivalent formulation in terms of a lagrangian functional extremization and, as a starting point, we decided to develop our analysis through two simplified problems:

- Static plasma behaviour (classical Grad-Shafranov);
- Non-magnetic flow behaviour (hydrodynamic).

The first case has been widely studied and the differential approach yields

$$-\triangle^{*}\psi = 4\pi r^{2}\frac{dp}{d\psi} + I\frac{dI}{d\psi}, \tag{8}$$

where $p = p(\psi)$ and $rB_{\phi} = I(\psi)$ are the two field-lines constants of the problem. In variational term the solution of the above expression is equivalently achieved finding an extremum of the lagrangian functional $L(\psi) = \int_{\Omega}\mathscr{L}(\mathbf{x},\psi,\nabla\psi)\,dV$, where the lagrangian density $\mathscr{L}$ is defined as follows

$$\mathscr{L} = -\frac{1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^{2} + \frac{1}{8\pi}\left(\frac{I}{r}\right)^{2} + p. \tag{9}$$

Note that in this case the langrangian of the system is a functional of the flux function only and that the lagrangian density can also be written as $\mathscr{L} = -\frac{B_P^2}{8\pi} + \frac{B_\phi^2}{8\pi} + p$.

In the hydrodynamic case there are three flux invariants: introducing the specific angular momentum per unit mass about the symmetry axis, $A = rv_{\phi}$, and the *Bernoulli 'constant'*

$$B = \frac{1}{2}v_P^2 + \frac{1}{2}\frac{A^2}{r^2} + \int (dp/\rho)\Big|_{\psi=const}, \tag{10}$$

it is then easy to prove that $B = B(\psi)$, $A = A(\psi)$ and $S = S(\psi)$. From the equation of state also follows $p = C(\psi)\rho^{\gamma} = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}\exp\left[(\gamma-1)\frac{S(\psi)}{k_B}\right]$.

From the momentum equation we obtain a second order partial differential equation

$$\left[r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial z}\right)\right] = \rho r^{2}\frac{dB}{d\psi} - \rho A\frac{dA}{d\psi} - r^{2}\frac{p}{k_B}\frac{dS}{d\psi}, \tag{11}$$

with ρ determined by the algebraic Bernoulli equation

$$B(\psi) = \frac{1}{2}\left(\frac{1}{r}\frac{\nabla\psi}{\rho}\right)^{2} + \frac{1}{2}\left[\frac{A(\psi)}{r}\right]^{2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1}C(\psi). \tag{12}$$

A solution of eq. (11) with the implicit condition stated by eq. (12) is found taking ρ and ψ such that the lagrangian of the system is an extremum of the functional

$$L(\rho,\psi) = \int_{\Omega}\mathscr{L}(\mathbf{x},\rho,\psi,\nabla\psi)\,dV, \tag{13}$$

where $\mathscr{L}$, the lagrangian density, is defined by

$$\mathscr{L} = \frac{1}{2}\rho\left(\frac{1}{r}\frac{\nabla\psi}{\rho}\right)^2 - \frac{1}{2}\rho\left(\frac{A}{r}\right)^2 + \rho B - \frac{1}{\gamma-1}\rho^\gamma C. \tag{14}$$

We notice that this last expression can be regarded as $\mathscr{L} = \rho v_P^2 + p$ and that the lagrangian functional depends on two independent variables.

Finally, for the Grad-Shafranov generalized equation, we can find a solution that also satisfies the generalized Bernoulli constraint just extremizing the lagrangian functional

$$L(\rho,\psi) = \int_\Omega \mathscr{L}(\mathbf{x},\rho,\psi,\nabla\psi)\,dV, \tag{15}$$

where $\mathscr{L}$ is now defined by

$$\mathscr{L} = \left(\frac{F^2}{4\pi\rho} - 1\right)\frac{1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^2 + \frac{1}{8\pi}\left(\frac{H + rv_\phi F}{r}\right)^2 - \frac{1}{2}\rho v_\phi^2 + \rho\left(J + rv_\phi G\right) - \frac{1}{\gamma-1}\rho^\gamma C. \tag{16}$$

As expected, the lagrangian density can be written in the form $\mathscr{L} = -\frac{B_p^2}{4\pi} + \frac{B^2}{8\pi} + p + \rho v_p^2$ and the Bernoulli constraint implies a double functional dependence for the Lagrangian. This variational formulation can be obtained from an energy principle with the flux functions (4-6) imposed as lagrangian multipliers (see ref[3]).

In the study of the functional (15) we have considered that the boundary conditions are the values of the unknown function ψ assigned on $\partial\Omega$. Another classical type of boundary condition that results very important for the variational problem we are attempting to investigate is obtained when the values assumed by ψ on a part $\overline{\partial\Omega}$ of the whole boundary $\partial\Omega$ are not specified. This condition is called *natural* and yield a solution that satisfies the subsequent equation on $\overline{\partial\Omega}$

$$\frac{\partial\mathscr{L}}{\partial\left(\partial\psi/\partial r\right)}\frac{dz}{ds} - \frac{\partial\mathscr{L}}{\partial\left(\partial\psi/\partial z\right)}\frac{dr}{ds} = 0, \tag{17}$$

where s is the boundary arc-length. In fact, taking the first variation of our functional, we obtain with some manipulations:

$$\delta L = \int_\Omega [\mathscr{L}]_\psi\,\delta\psi dV + \int_\Omega [\mathscr{L}]_\rho\,\delta\rho dV + \int_{\overline{\partial\Omega}}\left(\mathscr{L}_{\psi_z}\frac{dr}{ds} - \mathscr{L}_{\psi_r}\frac{dz}{ds}\right)\delta\psi dS$$

and, for the arbitrariness of the variations $\delta\psi$ and $\delta\rho$, the search for an extremum of L implies both the Euler-Lagrange equations $[\mathscr{L}]_\psi$, $[\mathscr{L}]_\rho$ and the equation (17) for the boundary $\overline{\partial\Omega}$ to be satisfied. Substituting $\mathscr{L}$ in (17) and carrying out the calculation, it can be rewritten in the form: $\frac{1}{r^2}\left(\frac{F^2}{4\pi\rho} - 1\right)\nabla\psi\cdot\mathbf{n} = 0$. This relation definitely means that in our case, in the part of $\partial\Omega$ where we impose no condition on ψ, we are asking for a solution with a poloidal velocity field flowing perpendicularly to the boundary.

We search for a simple approximation of ψ and ρ through the extremization of the functional (15), for this reason we start assuming that the approximation functions belongs to a finite dimensional subspace of the solution space, thus considering its expansion as a sum of base-function, it can be written in the form:

$$\psi(x,y) = \sum_{n=0}^{n_t} \psi_n F_n(x,y) = \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} \psi_{ij} F_i^x(x) F_j^y(y), \tag{18}$$

$$\rho(x,y) = \sum_{m=0}^{m_t} \rho_m G_m(x,y) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} \rho_{ij} G_i^x(x) G_j^y(y), \tag{19}$$

where $F^{x,y}$ and $G^{x,y}$ are two family of base-functions. It is usually preferred to approximate the solution with smooth functions, choosing the base that best represents the solution as we suppose it will be. It is also important to avoid singularities of the integral functions. First, we impose the boundary constraints of the problem. Substitution of the obtained functions into (13) implies

$$L(\psi_1,\ldots\psi_{n_l},\rho_1,\ldots,\rho_{m_t}) = \int_\Omega \mathscr{L}(\mathbf{x},\rho(\rho_m),\psi(\psi_n),\nabla\psi(\psi_n))\,dV, \tag{20}$$

and the extremization process can be developed deriving (20) with respect to ψ_n and ρ_m

$$\frac{\partial \mathscr{L}}{\partial \psi_n} = \int_\Omega \frac{\partial \mathscr{L}}{\partial \psi_n} dV = P_n(\psi_1,\ldots\psi_{n_l},\rho_1,\ldots,\rho_{m_t}), \tag{21}$$

$$\frac{\partial \mathscr{L}}{\partial \rho_m} = \int_\Omega \frac{\partial \mathscr{L}}{\partial \rho_m} dV = R_m(\psi_1,\ldots\psi_{n_l},\rho_1,\ldots,\rho_{m_t}). \tag{22}$$

Hence we conclude that the approximation coeficients are a solution of the non-linear equation system $[P_1,\ldots,P_{n_l},R_1,\ldots,R_{m_t}] = 0$. Since each term of this non-linear equation's system is a smooth function of unknown coefficients, we decide to exploit the Newton-Raphson's algorithm. Except for flow regimes with discontinuities (i.e. shock waves or transonic points), the method proposed has turned out to be accurate in describing general flows properties, proving itself to be a reliable instrument for the study of axial symmetric MHD flows.

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