Critical exponents for structural transitions in a complex plasma

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Introduction

The theory of critical phenomena has mostly been explored from the perspective of the statistical thermodynamics. In so-called extensive systems, the number of interacting particles is of the order of Avogadro's number, so the assumption of an infinite uniform system is justified. In non-extensive systems where the number of particles is of the order of fewer than 10^3, the thermodynamic limit can no longer be assumed, since the extent of the interparticle interaction is comparable to the size of the system. Complex plasma provides an ideal medium for studying phase transitions in non-extensive systems, with as little as two particles displaying extremely rich physics such as spontaneous symmetry-breaking and universality.

The critical instability of two dust particles levitating in the complex plasma sheath of a radio-frequency discharge is considered. It is shown that the two-particle system has a critical point where the alignment symmetry is continuously broken as the system parameter is varied. The associated critical exponents are derived and found to belong to the Ising universality class. Another universality class is suggested for symmetry breaking of the confinement in the horizontal and vertical directions.

Results

The Order Parameter Exponent β

Below a critical radial confinement $\omega_{\rho,\mathrm{c}}$. The Hamiltonian has a stable equilibrium with the dust particles horizontally aligned ($\Delta z = 0$). Taylor expanding the Hamiltonian about the equilibrium position in the vertical interparticle separation order parameter Δz it can be shown that asymptotically close to the critical point

$$\mathscr{H} = -\frac{1}{2}M\omega_z^2\zeta\Delta z^2 + \frac{1}{4}a_4\Delta z^4 \tag{1}$$

where $\zeta \equiv \frac{\omega_\rho - \omega_{\rho,\mathrm{c}}}{\omega_{\rho,\mathrm{c}}}$ and a_4 depends on the internal parameters of the dust system. Equation (1) corresponds to the Ginzburg-Landau Hamiltonian for the Ising model in the mean field approximation with zero external magnetic field strength $H = 0$ [1]. The equilibria of (1) are found using the minimum condition $\mathrm{d}\mathscr{H}/\mathrm{d}\Delta z = 0$, that is $0 = -M\omega_z^2\zeta\Delta z + a_4\Delta z^3$. For $\zeta < 0$, the only real solution is $\Delta z = 0$. That is, below the critical frequency, the dust particles remain aligned in the horizontal plane. Above the critical frequency $\zeta \geq 0$, the ground-state of the system bifurcates into two degenerate equilibria which are related by the $\Delta z \to -\Delta z$ symmetry of

the Hamiltonian $\Delta z = \pm \sqrt{\frac{M \omega_z^2 \zeta}{a_4}}$. The order parameter changes continuously as the frequency passes the critical point, with the critical exponent $\beta = 1/2$ defined by $\Delta z \propto |\zeta|^\beta$ characteristic of a continuous, or second-order phase transition. The Ising model of ferromagnetics undergoes such a transition from paramagnetic to ferromagnetic phase as $T \to T_c^+$ at zero external magnetic field strength. In this case, with the zero-field magnetisation $M|_{H=0} \propto |\varepsilon|^{1/2}$ as the order parameter. At non-zero magnetic field strength, the system loses its spin-reversal symmetry, so that the continuous phase transition can no longer occur. The external magnetic field H is said to be conjugated to the order parameter M.

The Response Exponent δ

Motivated by the external magnetic field of the Ising model, we search for a field conjugated to the order parameter Δz of the dust system. The asymmetric wake provides this field. Introducing an asymmetry between the wake charges of two particles ΔQ_w induces explicit symmetry breaking terms in the Hamiltonian

$$\mathscr{H} = a_1 \Delta Q_w \Delta z - \frac{1}{2} M \omega_z^2 \zeta \Delta z^2 + \frac{1}{3} a_3 \Delta Q_w \Delta z^3 + \frac{1}{4} a_4 \Delta z^4. \tag{2}$$

Note that it is necessary to replace each instance of Q_w in the expressions for the coefficients a_1 and a_4 by the sum of both wake charges, which is assumed to remain constant. The symmetry breaking terms skew the Hamiltonian so that the oblique equilibria lose their degeneracy and one becomes energetically favoured.

Along the critical isofrequency $\zeta = 0$, the equilibrium condition gives $\Delta Q_w = -\frac{a_4 \Delta z^3}{a_1 + a_3 \Delta z^2}$. The first term in the Taylor expansion of ΔQ_w about $\Delta z = 0$ occurs at third order in Δz, and all subsequent terms occur at fifth order or higher. This provides the critical exponent $\delta = 3$ for the scaling relation $\Delta z \propto \Delta Q_w^{1/\delta}$. If the wake field ΔQ_w passes zero at or above the critical frequency, then the dust system will change continously from one oblique equilibrium to the other. If this occurs below the critical frequency, the order parameter Δz develops a singularity at $\Delta Q_w = 0$, resulting in a discontinuous jump in the order parameter, known as a first order phase transition.

The Susceptibility Exponent γ

The susceptibility χ of a system is defined as the linear response of the order parameter to infinitesimal changes in the conjugate field. The critical exponent γ is defined such that $\chi = (\partial M / \partial H)_{H \to 0} \propto |\varepsilon|^{-\gamma}$. In the mean-field approximation, the predicted value is $\gamma = 1$

The equilibria of the general Hamiltonian of the dust system (2) are analytically soluble. Differentiating the solution with respect to the wake difference ΔQ_w and taking the limit of the derivative as $\Delta Q_w \to 0$ we obtain the critical exponent $\gamma = 1$ for the susceptibility

$$\chi = (\partial \Delta z / \partial \Delta Q_{\text{w}})_{\Delta Q_{\text{w}} \to 0} \propto |\zeta|^{-\gamma}.$$

Confinement Symmetry Breaking

We now consider symmetry breaking of a different kind, namely breaking of confinement symmetry in the radial and vertical directions. The vertical ion flow in the sheath naturally provides a prevalent direction, reducing the symmetry of the system. In order to describe the extent of this symmetry breaking, we use the inverse critical frequency ratio $\Psi \equiv \Omega^{-1}$ as an order parameter. Linear perturbation theory suggests the following expression for the critical ratio

$$\Omega^2 = 1 + \frac{Q_{\text{w}}Q_{\text{d}}}{2\varepsilon_0 M \omega_\rho^2} \left[\frac{\ell^2 \kappa}{\Delta_{\text{w}}^4} - (1 + \kappa \Delta_{\text{w}}) \left(\frac{3\ell^2}{\Delta_{\text{w}}^5} + \frac{\ell^2 \kappa}{\Delta_{\text{w}}^4} \right) \right] e^{-\kappa \Delta_{\text{w}}} \tag{3}$$

where Δ_{w} is the separation between each particle and the wake-field of the adjacent particle. This is a generalisation of a previous result [2] to include the effects of linear electron screening. Rearranging (3) to eliminate ω_ρ explicitly from the right-hand side we obtain

$$\Psi^2 = 1 - \frac{Q_{\text{w}}Q_{\text{d}}}{2\varepsilon_0 M \omega_z^2} \left[\frac{\ell^2 \kappa}{\Delta_{\text{w}}^4} - (1 + \kappa \Delta_{\text{w}}) \left(\frac{3\ell^2}{\Delta_{\text{w}}^5} + \frac{\ell^2 \kappa}{\Delta_{\text{w}}^4} \right) \right] e^{-\kappa \Delta_{\text{w}}}. \tag{4}$$

Plotting Ψ as a function of Q_{w} at constant ω_z reveals a curve which terminates at a critical wake charge $Q_{\text{w,c}}$. Given that the coordinates of the critical point in the (Q_{w}, Ψ) space are $(Q_{\text{w,c}}, 0)$, it quickly follows that near the critical point $\Psi = \pm\sqrt{\frac{Q_{\text{w,c}} - Q_{\text{w}}}{Q_{\text{w,c}}}}$ and thus we obtain the critical exponent $\beta = 1/2$ defined by $\Psi \sim (-\theta)^\beta$, where $\theta = (Q_{\text{w}} - Q_{\text{w,c}})/Q_{\text{w,c}}$.

Critical exponents can also be defined for the "spacial susceptibility" $\gamma = 1/2$ and the response exponent $\delta = 2$. The set $\{\beta = 1/2, \gamma = 1/2, \delta = 2\}$ satisfies the Widom equality

$$\gamma = \beta(\delta - 1)$$

suggestive of universality. The Griffith's equality

$$\alpha + \beta(\delta + 1) = 2$$

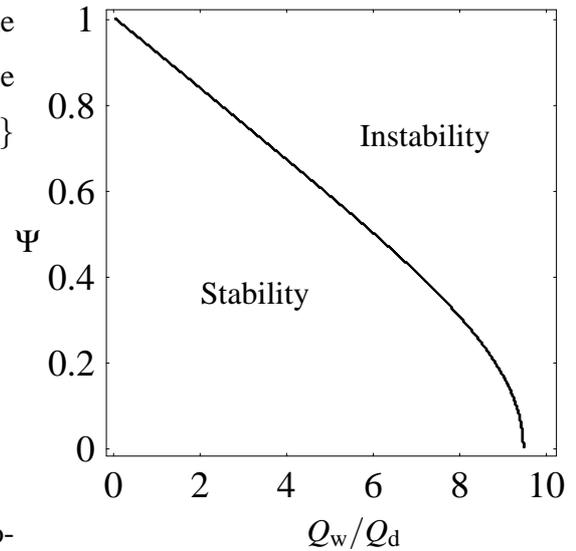can then be used to predict the specific heat exponent $\alpha = 1/2$.

Figure 1: The numerical phase diagram ($\lambda_{\text{D}} = 6.07 \times 10{-}4$, $M = 3.60 \times 10{-}14$, $Q_{\text{d}} = 3.6 \times 103e$, $\ell = 1.5\lambda_{\text{D}}$)

Discussion and Conclusion

The critical exponents for the horizontal alignment instability are the same as those of the mean field theory for thermodynamic systems such as the Ising model and Van der Waals theory. The dust system exhibits both a continuous (second order) phase transition at the critical resonant frequency, as well as discontinuous phase transitions induced by asymmetric wake fields. The exponents are independent of the plasma parameters such as the Debye length. Although we made use of the a the point-charge approximation to model the ion wake distribution, the results should apply equally well for any wake distribution since they depend only on the local approximation to wake potential up to third order.

In terms of spatial symmetry breaking of the confinement fields, the critical frequency ratio Ψ for two identical dust particles provides an order parameter to describe the extent of the symmetry breaking by the dust-induced wake fields. The Widom equality gives strong support to the notion of universality. It is not clear, however, if the predicted exponent of $\alpha = 1/2$ can be derived from the Hamiltonian.

The fact that the horizontal alignment instability belongs to the Ising universality class for thermodynamic systems suggests a common symmetry underlying these seemingly disparate systems. The new universality class for breaking of the external confinement symmetry needs to be explored in greater detail so that it can be compared to other systems which share the same symmetry.

References

[1] L. D. Landau and E. M. Lifschitz, *Lehrbuch der theoretischen Physik* (Band V, Akademie, Berlin, 1966).

[2] V. Steinberg, R. Sütterlin, A. V. Ivlev, and G. Morfill, *Phys. Rev. Lett.*, **86**, 4540 (2001).