

## Magnetic turbulent transport from self-consistent action-angle theory

R. Gatto, I. Chavdarovski

*Dept. of Physics, University of Rome "Tor Vergata", Rome, Italy*

**Introduction.** Understanding electron transport in magnetized fusion plasmas is essential to explain the evolution of present-days confinement experiments, and to envisage ways to improve their performances. Among the relevant issues awaiting for a clear explanation are (i) the presence of peaked density profiles in the absence of core sources, and (ii) the influence of plasma turbulence on the evolution of the current density. Recent advances in these two areas have led to the formulation of turbulent transport models which possess a particle pinch driven by temperature gradient and/or safety-factor gradient [2], and of turbulent versions of Ohm's law containing hyper-resistive and anomalous-resistivity terms.

We address the problem of particle, energy and current density transport employing the self-consistent (SC) action-angle collision operator [3, 4], which extends the original quasi-linear (QL) collision operator proposed by Kaufman [5] by including the important contribution associated with the particle back-reaction on the fluctuations. Action-angle variables  $(J, \Theta)$  are convenient phase-space variables in that the actions are adiabatic invariants which change only because of resonant wave-particle interactions. As a consequence the unperturbed motion is trivial,  $\Theta$  developing linearly in time with frequency  $\Omega = \dot{\Theta}$ , and the particle Hamiltonian can be written as  $H(J, \Theta) = H_0(J) + h(J, \Theta; t) + \dots$ , where the first order perturbation can be conveniently expanded in a Fourier series in the ignorable and periodic coordinates,  $h(J, \Theta; t) = \sum_{\ell} h(\ell, J; t) \exp(i\ell \cdot \Theta)$ . The triplet of integers  $\ell = (\ell_g, \ell_b, \ell_{\zeta})$  singles out each one of the harmonics of the particle perturbing Hamiltonian or, analogously, of the orbital motion. For a tokamak plasma the corresponding actions are chosen to be the gyro-action  $J_g$ , the toroidal angular momentum  $J_{\zeta}$  and the bounce action  $J_b$ . While this framework leads to important mathematical simplifications in the kinetic formulation of the transport problem, it has the drawback that the connection between action-angle diffusion and flux-surface-averaged radial transport is not trivial. Important steps have however already been taken in the direction of deriving from action space collision operators explicit transport models. A procedure for deriving flux-surface-averaged radial transport equations from the QL collision operator, presented in Ref. [6], has been used in Ref. [7] to derive the QL transport equations in the presence of low-frequency electrostatic fluctuations. Successively, the limitation intrinsic in the QL approach have been overcome in Refs [8, 9, 10], where a series of transport issues have been studied in the framework of the SC action-angle theory. In the present work we use the SC action-angle

collision operator to study axisymmetric, passing-electron transport induced by weak or moderate steady-state magnetic turbulence. We advance the present status by presenting the explicit form of the complete transport matrix, including the turbulent Ohm's law.

**Theoretical framework.** The Fokker-Planck equation in action space for the zero-order part of the  $\Theta$ -averaged distribution function of scattered species 1 is

$$\frac{\partial f(J_1; t)}{\partial t} = \frac{\partial}{\partial J_1} \cdot \left[ \mathbf{D}(J_1) \cdot \frac{\partial f(J_1; t)}{\partial J_1} - \mathbf{F}(J_1) f(J_1; t) \right].$$

Here,  $\mathbf{D}(J; t)$  is the usual (ensemble averaged) diffusion tensor, and  $\mathbf{F}(J, t)$  is the friction vector which considers the polarization field induced by the test particle. The latter term, neglected in the quasilinear approach, is necessary for achieving a self-consistent theory. If we specialize to tokamak geometry and assume a drifting, locally Maxwellian distribution function  $f_M$ , the radial flux of the quantity  $\chi = 1, Mv_{\parallel}, Mv^2/2$  at location  $\alpha = \bar{\alpha}$  [where  $\alpha$  is the toroidal flux function, related to the cylindrical radius by  $r = (2\alpha/B_{0,t})^{1/2}$ ] can be expressed as follows:

$$\begin{aligned} (\mathbf{I} \cdot \nabla \bar{\alpha})_1^\chi &= \sum_2 \sum_{\ell_1, \ell_2} \frac{1}{\mathcal{V}'} \left( \frac{2\pi}{M_1} \right)^3 \int dJ_1 \left( \frac{2\pi}{M_2} \right)^3 \int dJ_2 \\ &\quad Q(\ell_1, J_1; \ell_2, J_2) X^\chi(J_1, t, \ell_1, \bar{\alpha}) f_M(J_1) f_M(J_2) \mathcal{A}(J_1, J_2, \ell_1, \ell_2), \end{aligned} \quad (1)$$

where the driving term is

$$\begin{aligned} \mathcal{A}(J_1, J_2, \ell_1, \ell_2) &\equiv \left( 1 + \frac{V_{\parallel,1} P_1}{T_1} \right) \left( 1 + \frac{V_{\parallel,2} P_2}{T_2} \right) \left( \frac{\ell_1 \cdot \Omega_1}{T_1} - \frac{\ell_2 \cdot \Omega_2}{T_2} \right) \\ &- \left[ \left( 1 + \frac{V_{\parallel,2} P_2}{T_2} \right) G_1 \frac{V_{\parallel,1}}{T_1} - \left( 1 + \frac{V_{\parallel,1} P_1}{T_1} \right) G_2 \frac{V_{\parallel,2}}{T_2} \right] - \left( 1 + \frac{V_{\parallel,1} P_1}{T_1} \right) \left( 1 + \frac{V_{\parallel,2} P_2}{T_2} \right) (g_1 \mathcal{A}_{N,1} - g_2 \mathcal{A}_{N,2}) \\ &\quad - \left[ \frac{V_{\parallel,1} P_1}{T_1} \left( 1 + \frac{V_{\parallel,2} P_2}{T_2} \right) g_1 \mathcal{A}_{V,1} - \frac{V_{\parallel,2} P_2}{T_2} \left( 1 + \frac{V_{\parallel,1} P_1}{T_1} \right) g_2 \mathcal{A}_{V,2} \right] \\ &\quad - \left( 1 + \frac{V_{\parallel,1} P_1}{T_1} \right) \left( 1 + \frac{V_{\parallel,2} P_2}{T_2} \right) \left( g_1 \frac{K_{0,1}}{T_1} \mathcal{A}_{T,1} - g_2 \frac{K_{0,2}}{T_2} \mathcal{A}_{T,2} \right), \end{aligned} \quad (2)$$

$K_0 \equiv Mv^2/2$ ,  $P \equiv Mv_{\parallel}$ , and where

$$g \equiv \ell \cdot \nabla_J \alpha(J), \quad G \equiv \ell \cdot \nabla_J P[\alpha(J)], \quad X^\chi \equiv \int \frac{d\Theta}{(2\pi)^3} \chi g \delta(\alpha - \bar{\alpha}), \quad (3)$$

$$\mathcal{A}_N \equiv \frac{N'}{N} + \frac{q}{T} \Phi'_0 - \frac{3}{2} \frac{T'}{T}, \quad \mathcal{A}_T \equiv \frac{T'}{T}, \quad \mathcal{A}_V \equiv \frac{V'_{\parallel}}{V_{\parallel}} - \frac{T'}{T}. \quad (4)$$

Eq. (1) with definitions (2)-(4) is the radial flux of  $\chi$  of the scattered particles (species 1) due to the scattering particle distribution (species 2). The kernel  $Q$  which expresses the fluctuation spectrum driving transport is given by:

$$Q(\ell_1, J_1, \ell_2, J_2) \equiv 2\pi \delta(\ell_1 \cdot \Omega_1 - \omega) \sum_a |C_a(\ell_1, J_1, \ell_2, J_2, \omega)|^2 \Big|_{\omega=\omega_2 \equiv \ell_2 \cdot \Omega_2},$$

where the coupling coefficient is  $C_a(\ell_1, J_1, \ell_2, J_2, \omega) \equiv [4\pi h_a(\ell_1, J_1, \omega) h_a^*(\ell_2, J_2, \omega)]/[N_a \Delta_a(\omega)]$ . Here  $N_a$  is a normalizing factor, and  $\Delta_a$  is the eigenvalue of the Maxwell operator, relative to normal mode “ $a$ ”.

**Flux evaluation.** To evaluate Eq. (1) we proceed as follows. First, following Ref. [8], we express  $Q$  in terms of the pseudo-thermal ansatz, an approximation of a magnetic turbulent spectrum which retains the thermal structure (thus preserving the symmetries of the collision operator) but replaces its form so to better represent experimental features. The various factors in Eq. (1) are then evaluated in the far-untrapped limit, following procedures worked out in Ref. [6] but extended to include the radial dependence of the safety factor. Ignoring gyro-resonances and neglecting contributions from the gradient of the equilibrium electrostatic potential, we obtain for the factors in Eq. (3) the expressions

$$g \simeq \frac{c}{q} \ell_b, \quad G \simeq \frac{n_a v_{\parallel} M c}{q r B_{0,t}} \frac{d q_{\text{saf}}}{d r} + \frac{1}{R_0} \left( \frac{\ell_b}{q_{\text{saf}}} + n_a \right), \quad \ell \cdot \Omega \simeq v_{\parallel} G, \quad X^{\chi} \simeq \frac{c}{q} \ell_b \chi \delta(\alpha - \bar{\alpha}),$$

$n_a$  being the familiar toroidal mode number. These expressions show that the terms in Eq. (2) proportional to the factors  $G$  and  $\ell \cdot \Omega$  include drives proportional to the safety factor gradient.

**Results.** We perform the  $\ell$  summations in the  $M_1 \ll M_2$  approximation, again following Ref. [8], and carry out the  $\mathbf{k}$  summations assuming magnetic turbulence with  $k_{\parallel} \ll k_{\perp}$ . As a final step we perform the action-space integrations by transforming them in integrations over the variables  $\alpha$ ,  $v_{\parallel}$  and  $K_0$ , as suggested in Ref. [7]. Following this procedure we obtain the following expression for the electron-ion particle flux:

$$(\mathbf{I} \cdot \nabla \bar{\alpha})_{ei}^N = -\mathcal{L}_{ei}^N \left( \mathcal{A}_{Ne} + \frac{5}{2} \mathcal{A}_{Te} + \mathcal{A}_{Ni} + \frac{5}{2} \mathcal{A}_{Ti} + 3 \mathcal{A}_{q_{\text{saf}}} \right),$$

where the last term, driven by the safety factor gradient scale-length  $\mathcal{A}_{q_{\text{saf}}} = (1/q_{\text{saf}})(dq_{\text{saf}}/dr)$ , is a pinch. The transport coefficient is found to be  $\mathcal{L}_{ei}^N = \pi q_{\text{saf}} N_e \hat{D}^{RR}$  where  $\hat{D}^{RR}$  is the Rechester Rosenbluth transport coefficient  $R_0 v_{th,e} \tilde{b}^2(i)$ , possibly generalized to include FLR and drift effects. Analogously, the electron thermal energy balance  $(\mathbf{I} \cdot \nabla \bar{\alpha})_{ei}^T = -\chi_N T_e (dN_e/dr) - \chi_T N_e (dT_e/dr) - (5/2) T_e (\mathbf{I} \cdot \nabla \bar{\alpha})_{ei}^N + V_T N_e T_e$  is characterized by the thermal diffusivity  $\chi_T = (15/2)(\mathcal{L}_{ei}^N/N_e)$  and by the energy pinch velocity

$$V_T = -5 \frac{\mathcal{L}_{ei}^N}{N_e} \left( \frac{1}{N_i} \frac{dN_i}{dr} + \frac{1}{T_i} \frac{dT_i}{dr} + 6 \mathcal{A}_{q_{\text{saf}}} \right).$$

The outward energy flux induced by the ion's profiles is contrasted by the inward component due to the safety factor gradient. To derive Ohm's law, we introduce the approximate expression

$V_e \simeq -J_{\parallel}/(eN_e)$  in the electron momentum balance. We obtain  $-\frac{M_e}{e^2 N_e} \frac{\partial \langle J_{\parallel} \rangle_r}{\partial t} + \frac{\langle N_e E_t \rangle_r}{N_e} = E_{\text{BS}} + \eta_{\parallel}^{\text{neo}} J_{\parallel} + E_{\parallel}^{\text{SC}}$ , where on the right-hand side we have added the bootstrap and neoclassical resistive terms, and where the new self-consistent contribution is

$$E_{\parallel}^{\text{SC}} = -\frac{1}{N_e r} \frac{d}{dr} r \left[ \eta \left( \mathcal{A}_{N_e} - \frac{1}{L_e} + \frac{5}{2} \mathcal{A}_{T_e} + \mathcal{A}_{N_i} + \frac{5}{2} \mathcal{A}_{T_i} + 4 \mathcal{A}_{q_{\text{saf}}} \right) J_{\parallel} + \eta \frac{dJ_{\parallel}}{dr} \right],$$

where  $\eta \equiv 3M_e \mathcal{L}_{ei}^N / (e^2 N_e)$  and  $L_e \equiv [1/N_e (dN_e/dr)]^{-1}$ . The term proportional to the current density gradient can be manipulated to give rise to a hyper-resistive term (and other terms),

$$\left( E_{\parallel}^{\text{SC}} \right)_{\text{hyper}} = -\frac{1}{B_0} \frac{1}{r} \frac{d}{dr} \left[ r \eta_H \frac{d}{dr} \left( \frac{J_{\parallel}}{B_0} \right) \right]$$

where  $\eta_H \equiv \eta B_0^2 / N_e$ . In conclusion, we have derived from the self-consistent action-angle collision operator the full set of transport fluxes relative to passing electrons in magnetic micro-turbulence. A pinch term proportional to the gradient of the safety factor is present in both the particle and the energy flux. Gradients in density and temperature (of both species) always lead to outward fluxes. The turbulent Ohm's law derived from the momentum balance includes terms proportional to the safety factor gradient, as well as to the density and temperature gradients, and a hyper-resistive term which accounts for the tendency of the turbulence to flatten the parallel current density profile.

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