

Unstable magnetohydrodynamical continuous spectrum of accretion disks

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Introduction

Accretion disks are very common objects in the universe. These disks can be found in stellar systems as well as in active galactic nuclei (AGN). In a stellar system, the disk accretes matter onto a protostar (young stellar object), white dwarf, neutron star, or a black hole. The typical size of these disks is a few hundred astronomical units (AU). In an AGN the central object is a massive black hole and the typical size of the disk is a hundred parsecs (pc).

Much of the research on accretion disk dynamics makes use of the MHD model. A lot of work studies waves and instabilities about a disk equilibrium in the cylindrical limit, which is essentially a one-dimensional model.

In this paper, we focus on two-dimensional MHD disk equilibria with purely toroidal flow. We present the equations and their numerical solutions of these kind of equilibria. For the numerical calculations we make use of the code FINESSE [1]. Next, we present a detailed stability study of these disk equilibria. The analysis is done theoretically as well numerically. From theory we show that a new type of instability, the convective continuum instability, may occur inside a disk. We also derive a necessary condition for the instability. The numerical part is done by making use of the spectral code PHOENIX [3].

Accretion disk equilibrium

The plasma inside a tokamak or accretion disk is modeled by making use of the ideal MHD equations. The equilibrium itself is assumed to be time-independent and axisymmetric. This time-independence is justified if we look at a timescale which much shorter than the dynamical timescale of the disk equilibrium. Because of the axisymmetry we make use of cylindrical coordinates (R, Z, φ) (in this order!), where R is the distance to the symmetry axis.

The magnetic field can be expressed in terms of the poloidal magnetic flux function ψ ,

$$\mathbf{B} = \frac{1}{R} \mathbf{e}_\varphi \times \nabla \psi + B_\varphi \mathbf{e}_\varphi. \quad (1)$$

We will consider disk equilibria with only toroidal flow, $\mathbf{v} = v_\varphi \mathbf{e}_\varphi = R\Omega \mathbf{e}_\varphi$. In this case the stationary MHD equations reduce to the extended Grad-Shafranov equation,

$$R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} = -\frac{1}{2} \frac{dI^2}{d\psi} - R^2 \frac{\partial p}{\partial \psi}, \quad (2)$$

where the poloidal stream function $I = I(\psi) = RB_\varphi$ and p is the pressure. Without toroidal flow and gravity we recover to the Grad-Shafranov equation [6], [12]. Along the poloidal magnetic field lines one has to solve two other equations,

$$\left. \frac{\partial p}{\partial R} \right|_{\psi=\text{const}} = \rho \left(R\Omega^2 - \frac{\partial \Phi}{\partial R} \right) \quad (3)$$

$$\left. \frac{\partial p}{\partial Z} \right|_{\psi=\text{const}} = -\rho \frac{\partial \Phi}{\partial Z}, \quad (4)$$

where Φ is the gravitational potential caused by for example the protostar or black hole. These latter two equations can be solved analytically if one assumes that the temperature, the entropy, or the density is a flux function. Here, we focus on the latter case, for which results that the pressure can be written as

$$p(\psi; R, Z) = p_0(\psi) \left[1 + (R^2 - R_0^2)\Omega^2 - \frac{\Phi}{T_\rho} \right], \quad (5)$$

where p_0 is the pressure for a static pure Grad-Shafranov equation without gravity, R_0 is the geometric axis of the accretion disk, and the quasi-temperature $T_\rho = p_0/\rho$.

Spectral formulation

The formalism introduced by Frieman and Rotenberg [4] is used to investigate the stability properties of stationary MHD equilibria. This formalism is best exploited in coordinates in which the magnetic field lines become straight line and we make use of Fourier representation, $\xi \propto \exp[i(n\varphi - \omega t)]$. Here, ξ is the Lagrangian displacement field and ω is the eigenfrequency. In this way the equation for the MHD continua where we concentrate on modes at fixed flux surfaces $\psi = \psi_0$ becomes

$$(\mathbf{a} + \mathbf{r} + 2\rho\tilde{\omega}\Omega\mathbf{c} - \rho\tilde{\omega}^2\mathbf{b}) \begin{pmatrix} \eta \\ \xi \end{pmatrix} = 0, \quad (6)$$

where the perpendicular and parallel component of the displacement field are $Y = \delta(\psi - \psi_0)\eta(\vartheta, \varphi)$ and $Z = \delta(\psi - \psi_0)\xi(\vartheta, \varphi)$, respectively. Here, matrices \mathbf{a} and \mathbf{b} correspond to those found for the static case without gravity. Matrix \mathbf{c} represents the Coriolis effect due to the rotation and matrix \mathbf{r} contains both toroidal flow and gravity contributions. Furthermore, $\tilde{\omega}$ is the Doppler shifted eigenfrequency. Explicit expressions are given in [2].

From the equation of the MHD continua (6) a necessary condition for the convective continuum instability (CCI) can be derived,

$$N_{\text{BV,pol}}^2 = - \left[\frac{\mathbf{B}_\vartheta \cdot \nabla p}{\rho B} \right] \left[\frac{\mathbf{B}_\vartheta \cdot \nabla S}{\gamma B S} \right] < 0, \quad (7)$$

where \mathbf{B}_ϑ , B , S , γ , and $N_{\text{BV,pol}}^2$ are the poloidal magnetic field, total magnetic field, ratio of specific heats, and the Brunt-Väisälä frequency projected on a flux surface. From this condition it is clear that equilibria with entropy $S = S(\psi)$ or equilibria with temperature $T = T(\psi)$, $\gamma \leq 1$ do not suffer from this instability. Equilibria where the density is a flux function may turn unstable. Notice that the necessary condition (7) has a close resemblance with the Schwarzschild criterion for convective instabilities. The only difference is that in the Schwarzschild criterion one deals with normal derivatives while in this criterion one deals with tangential ones. Hellsten and Spies [8], Hameiri [7], and Poedts et al. [9] noticed this also, in different astrophysical context.

Thick accretion disk

For the computation for a MHD equilibrium, the numerical code FINESSE [1] has been used. This finite element code uses a Picard iteration scheme to compute the solution of the extended Grad-Shafranov equation (2). The used elements are cubic to ensure sufficient accuracy needed for the MHD spectral analysis.

For modeling a thick accretion disk, the following flux functions have been used:

$$\begin{aligned} I^2(\psi) &= A(1 - 0.0385\psi + 0.02\psi^2 + 0.00045\psi^3), & \rho(\psi) &= 1 - 0.4\psi \\ p_0(\psi) &= AB(1 - 0.9\psi), & \Omega(\psi) &= C(1 - 0.9\psi^2), \end{aligned} \quad (8)$$

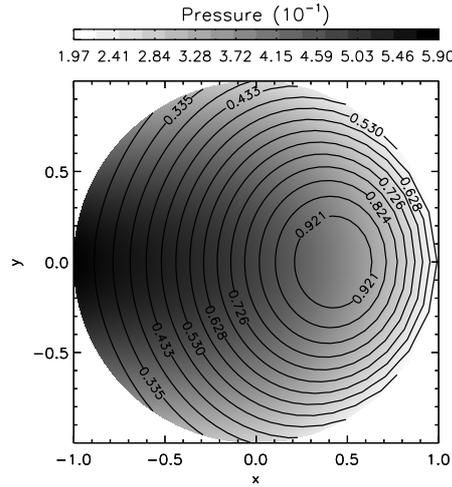


Figure 1: The pressure (gray-scale) and plasma beta $\beta = 2p/B^2$ (contours) for a thick accretion disk with density $\rho \propto (1 - 0.4\psi)$ and an inverse aspect ratio $\varepsilon = 0.5$. The central object is to the left of this figure.

where the coefficients A , B , and C are given by 91.5, 0.01, and 0.1. The poloidal cross-section is circular, the inverse aspect ratio $\varepsilon = a/R_0 = 0.5$. Here, a is the minor radius of the disk. The external gravitational potential is assumed to be a Newtonian one, i.e. $\Phi = -GM_*/\sqrt{R^2 + Z^2}$, with a scaled mass of the central object $GM_* = 1$. The two-dimensional pressure and plasma beta profile are shown in figure 1. The pressure decreases monotonically outwards due to the strong external gravitational potential. The temperature, not shown, also decreases in the outward direction. The plasma beta $\beta = 2p/B^2$ varies from 0.237 up to 0.970. The disk rotates totally sub-Keplerian which can be seen from the ratio $v_\phi/v_{\text{Kepler}} = [0.007, 0.282]$.

The MHD stability analysis has been performed with the code PHOENIX [3]. This code computes the complete MHD spectrum for a given two-dimensional equilibrium, toroidal mode number n , and a range of poloidal mode numbers m using the Jacobi-Davidson method [13]. A range of poloidal mode numbers is required due to the possible mode coupling caused by allowing for a non-circular cross-section, gravitational stratification, and toroidal geometry. PHOENIX solves the linearized MHD equations using a finite element method in the ψ -direction, a spectral method in the poloidal direction, and a standard Galerkin method is exploited to obtain a linear generalized eigenvalue problem. The used elements are a combination of quadratic and cubic Hermite elements to prevent the creation of spurious eigenvalues [11]. The continuous spectrum is computed by a less expensive method [10]. In this method the cubic and quadratic elements are replaced by $\log(\delta)$ and $1/\delta$ with $\delta \ll 1$, respectively. The resulting eigenvalue problem is then solved by a direct QR method per flux surface

The continuous MHD spectrum has been computed for this equilibrium. The toroidal and poloidal mode numbers have been chosen as $n = -1$ and $m = \{-3, \dots, 5\}$, respectively. Figure 2 contains the resulting eigenfrequencies in the complex plane. This figure shows clearly that the disk becomes unstable due to the presence of toroidal flow and gravity. This is in accord with the CCI instability criterion.

A more extensive study of this type of equilibria can be found in the article by Blokland et al. [2].

Conclusions

The equations which describe an axisymmetric MHD accretion disk equilibrium have been presented. The FINESSE code has been used to compute a numerical solution of these equations.

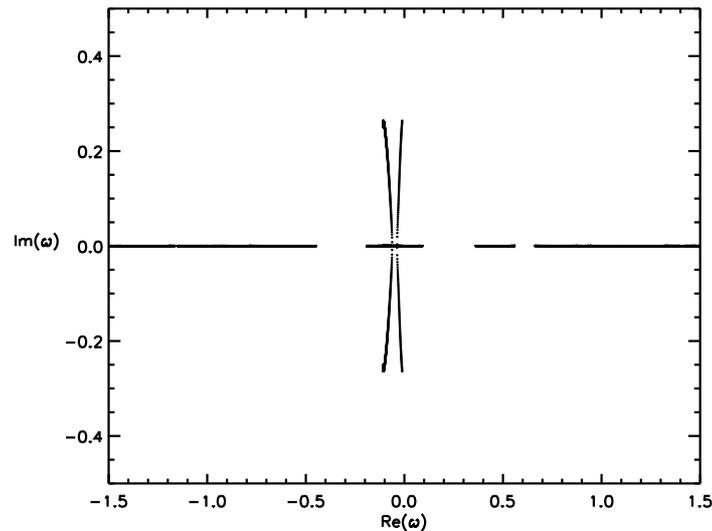


Figure 2: For the thick disk equilibrium shown in Fig. 1, the sub-spectrum of the MHD continua in the complex plane are shown for toroidal mode number $n = -1$ and poloidal mode numbers $m = [-3, 5]$.

We have considered a thick disk with a plasma beta of the order 0.1. The disk itself rotates totally sub-Keplerian. Furthermore, the numerical results show that the pressure can only decrease radially outward if there is a strong external gravitational potential.

The stability of the disk equilibria have been analyzed theoretically as well numerically. A necessary condition for the CCI has been derived from theory. This condition closely resembles the Schwarzschild condition. Only disks where the density is a flux function can be subject to the CCI. Our numerical results support this theoretical conclusion. These localized instabilities will lead to small-scale, turbulent dynamics when these kind of equilibria are used as initial condition for non-linear simulations. This dynamics is unrelated to the magneto-rotational instability. Indeed, these instabilities, as well as the poloidal flow driven unstable continuum modes investigated by Goedbloed et al [5], provide a new route to MHD turbulence in accretion disks.

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References

- [1] A.J.C. Beliën, M.A. Botchev, J.P. Goedbloed, B. van der Holst, and R. Keppens, *J. Comp. Physics* **182**, 91 (2002)
- [2] J.W.S. Blokland, R. Keppens, and J.P. Goedbloed, *A & A* **467**, 21 (2007)
- [3] J.W.S. Blokland, B. van der Holst, R. Keppens, and J.P. Goedbloed, To appear in *J. Comp. Physics* (2007)
- [4] E. Frieman, and M. Rotenberg, *Rev. Mod. Phys.* **32**, 898 (1960)
- [5] J.P. Goedbloed, A.J.C. Beliën, B. van der Holst, and R. Keppens, *Phys. Plasmas* **11**, 28 (2004)
- [6] H. Grad, and H. Rubin, *Proceedings of the second UN International Conf. on the Peaceful Uses of Atomic Energy* **31**, 190 (1958)
- [7] E. Hameiri, *Phys. Fluids* **26**, 30 (1983)
- [8] T.A.K. Hellsten, and G.O. Spies, *Phys. Fluids* **22**, 743 (1979)
- [9] S. Poedts, D. Hermans, and M. Goossens, *A & A* **151**, 16 (1985)
- [10] S. Poedts, and E. Schwarz, *J. Comp. Physics* **105**, 165 (1993)
- [11] J. Rappaz, *Numer. Math.* **28**, 15 (1977)
- [12] V.D. Shafranov, *Sov. Phys. JETP* **6**, 545 (1958)
- [13] G.L.G. Sleijpen, and H. van der Vorst, *SIAM J. Matrix Anal. Appli.* **17**, 401 (1996)