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The Ideal External Peeling Mode

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The potentially damaging ELMs in Tokamaks such as JET are believed to be triggered by the Peeling-Ballooning instability[1, 2]. However, numerical calculations[3] have suggested that the Peeling mode should be stable in plasmas whose separatrix forms an X-point, these include the plasmas in JET. In contrast, analytical work by Laval et al[4] concludes that the Peeling mode is unstable in cylindrical plasmas with a shaped cross-section that can in principle approach an X-point configuration; a conclusion that potentially conflicts with the numerical calculations. [1]

To help resolve these concerns, we consider the Peeling mode in toroidal tokamak geometry, generalising cylindrical calculations at marginal stability to arbitrary toroidal geometry. Our calculations solely concern the Peeling mode, and our results are not complicated by coupling to the Ballooning mode for example. We will relate these calculations to the $\delta W = \delta W_F + \delta W_S + \delta W_V$, of the energy principle[5], where δW_F is a plasma contribution, δW_S is a surface integral, and δW_V is a vacuum contribution. We find that the Peeling mode described by these studies, corresponds to taking $\delta W_F \ll \delta W_S + \delta W_V$, and solving (at marginal stability) $\delta W_S + \delta W_V = 0$. Our calculations do not use an expansion in a Fourier series, for example, and appear to be valid arbitrarily close to a separatrix with an X-point. Finally we consider Peeling mode stability in an X-point equilibrium, and explain the results found in Refs. [3] and [4].

We generalise studies of the Peeling mode at marginal stability in a circular cross-section plasma, by observing that they implicitly assume: (1.) Zero equilibrium skin currents, but the equilibrium current $\vec{J}_0$ can be discontinuous at the plasma-vacuum interface. (2.) Perturbations to the magnetic field induce perturbation skin currents. We will use these assumptions to solve the linearised form of the force balance equation $\vec{J} \wedge \vec{B} = \nabla p$ at marginal stability. We will write $\vec{B}_0 = I(\psi)\nabla\phi + \nabla\phi \wedge \nabla\psi$, for which

$$\begin{aligned} \frac{\vec{B}_p.\vec{J}_0}{B_p^2} &= -I'(\psi) \\ \nabla\phi.\vec{J}_0 &= -p'(\psi) - \frac{I(\psi)I'(\psi)}{R^2} \end{aligned} \tag{1}$$

Firstly we define perturbation skin currents as

$$\int_P^V \vec{J}_1 d\psi \equiv \vec{\sigma} \tag{2}$$

[1]The mode in Ref. [4] uses straight field-line variables for the poloidal angle, and hence can be difficult to represent in a numerical calculation when the outermost surface approaches a separatrix with an X-point.

with $\int_P^V$ denoting the integral from just inside the plasma to just inside the vacuum, with $\vec{\sigma}$ evaluated at the equilibrium surface position. We will also use Ampère's law and $\nabla.\vec{B} = 0$ at the surface, that require[6]

$$\left[\left|\vec{n}.\vec{B}\right|\right] = 0 \tag{3}$$

$$\nabla\psi \wedge \left(\vec{B}_V - \vec{B}\right) = \vec{\sigma} \tag{4}$$

where $[|f|]$ denotes the difference between f evaluated just outside the plasma minus the value of f evaluated just inside the plasma. Using $\vec{J}_1 = \nabla \wedge \vec{B}_1$, we may evaluate Eq. 2 in terms of the discontinuities in the perturbed magnetic field at the plasma surface, with

$$\vec{\sigma} = R^2 \nabla\phi [|\vec{B}_p.\vec{B}_1|] - R^2 \vec{B}_p [|\nabla\phi.\vec{B}_1|] \tag{5}$$

We take the curl of the linearised equation for $\nabla p = \vec{J} \wedge \vec{B}$, project in the poloidal, toroidal, and $\nabla\psi$ directions, and form $\int_P^V d\psi$, while noting that Eq. 5 implies $\nabla\psi.\vec{\sigma} = 0$; then we obtain

$$\begin{aligned} 0 &= \vec{B}_0.\nabla\left(\vec{\sigma}.\vec{B}_p\right) + B_1.\nabla\psi\left[\left|\vec{B}_p.\vec{J}_0\right|\right] - \vec{\sigma}.\nabla B_p^2 \\ 0 &= \vec{B}_0.\nabla\left(\vec{\sigma}.\nabla\phi\right) + B_1.\nabla\psi\left[\left|\nabla\phi.\vec{J}_0\right|\right] + 2I\frac{\vec{\sigma}.\nabla R}{R^3} \end{aligned} \tag{6}$$

We write $2I\frac{\sigma.\nabla R}{R^3} = -\sigma.\nabla\left(\frac{I}{R^2}\right) = -\left(\frac{\sigma.\vec{B}_p}{B_p^2}\right)\vec{B}_p.\nabla\left(\frac{I}{R^2}\right)$, and using Eqs. 1 we get,

$$\begin{aligned} 0 &= \vec{B}_0.\nabla\left(\frac{\vec{\sigma}.\vec{B}_p}{B_p^2} + I_a'\xi_\psi\right) \\ 0 &= \vec{B}_0.\nabla\left(\vec{\sigma}.\nabla\phi + \xi_\psi\left(p_a' + \frac{I_a I_a'}{R^2}\right)\right) - \vec{B}_0.\nabla\left(\frac{I}{R^2}\right)\left(\frac{\vec{\sigma}.\vec{B}_p}{B_p^2} + I_a'\xi_\psi\right) \end{aligned} \tag{7}$$

Therefore because $\vec{\sigma} = 0$ when $\xi_\psi = 0$, Eq. 7 implies that

$$\vec{\sigma} = \vec{J}_0 \xi_\psi \tag{8}$$

where we have used Eq. 1 to write Eq. 8 in terms of the equilibrium current $\vec{J}_0$ again.

Next we consider $[|\nabla.\vec{B}_0|]$, the difference between $\nabla.\vec{B}$ evaluated just inside the plasma and just outside the plasma, and consider the high-n toroidal mode number limit. Then using Eq. 5 we find,

$$R^2 B_p^2 \left[\left|\frac{i}{n}\frac{\partial}{\partial\psi}\nabla\psi.\vec{B}_1\right|\right] = \vec{B}_0.\vec{\sigma} + O\left(\frac{1}{n}\right) \tag{9}$$

This in conjunction with Eq. 8 is the generalisation of the Peeling mode's marginal stability condition at high-n to toroidal geometry. In the cylindrical limit it reduces to the usual result[2]

$$0 = \frac{rJ_\parallel}{B_p} + \frac{m - nq}{m}\left[\left|\frac{r\frac{db_r}{dr}}{b_r}\right|\right] \tag{10}$$

We outline how this result relates to the energy principle, starting from the high-n expression for δW given in Ref. [7]. We repeat here for convenience their result for δW_S:

$$\delta W_S = \pi \oint d\chi \frac{\xi_\psi^*}{n} J_\chi B k_\| \left[\frac{R^2 B_p^2}{J_\chi B^2} \frac{1}{n} \frac{\partial}{\partial \psi} \left(J_\chi B k_\| \xi_\psi \right) + \frac{\vec{B}.\vec{J}}{B^2} \xi_\psi \right] \tag{11}$$

with $J_\chi B k_\| = i J_\chi \vec{B}.\nabla$, and J_χ the Jacobian in the orthogonal χ, ψ, ϕ co-ordinate system.

In the vacuum $\nabla \wedge \vec{B}^V = 0$ so we may write $\vec{B}^V = \nabla V$, and $\nabla.\vec{B}^V = 0$ implies $\nabla^2 V = 0$. The boundary condition at the plasma-vacuum boundary requires[5]

$$\vec{n}_0.\vec{B}_1^V = \vec{B}_0.\nabla \left(\vec{n}_0.\vec{\xi} \right) - \left(\vec{n}_0.\vec{\xi} \right) \left(\vec{n}_0.\vec{n}_0.\nabla \vec{B}_0 \right) \tag{12}$$

which may be shown to be identical to requiring simply that $\nabla \psi.\vec{B}_1 = \nabla \psi.\vec{B}_1^V$, with both equations being evaluated at the equilibrium plasma surface. We also require that $\vec{B}^V \rightarrow 0$ at large distances from the plasma, ignoring the presence of a vacuum wall for simplicity.

We evaluate $\delta W_V = \frac{1}{2} \int \vec{dr} |\vec{B}_1^V|^2 = \frac{1}{2} \int \vec{dr} |\nabla V|^2$ by using Gauss' theorem and $\nabla^2 V = 0$ to write δW_V as a surface integral $\delta W_V = -\frac{1}{2} \int V \vec{n}_0.\nabla V^* dS$. The term $\vec{n}_0.\nabla V^* = \frac{\nabla \psi}{RB_p}.\vec{B}_1^{V*}$, which using the plasma vacuum boundary condition described above, $\nabla \psi.\vec{B}_1 = \nabla \psi.\vec{B}_1^V$, becomes $\vec{n}_0.\nabla V^* = \frac{1}{RB_p} \nabla \psi.\vec{B}_1^*$. The largest terms in a high-n expansion of $\nabla^2 V = 0$, require

$$V = \frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{\partial}{\partial \psi} \left(\nabla \psi.\vec{B}_1^V \right) \tag{13}$$

which along with $\vec{n}_0.\nabla V^* = \frac{1}{RB_p} \nabla \psi.\vec{B}_1^*$ and $dS = (J_\chi B_p) d\chi R d\phi$, may be substituted into δW_V. After integrating with respect to ϕ we obtain

$$\delta W_V = \pi \oint J_\chi d\chi \left(\frac{i}{n} \right) \frac{\nabla \psi.\vec{B}_1^*}{B^2} \left[R^2 B_p^2 \frac{i}{n} \frac{\partial}{\partial \psi} \left(\nabla \psi.\vec{B}_1 \right) \right] \tag{14}$$

Because $J_\chi B k_\| = i J_\chi \vec{B}.\nabla$, and for high-$n$ $\frac{1}{J_\chi} \frac{\partial}{\partial \psi} J_\chi \vec{B}.\nabla = \frac{\partial}{\partial \psi} \vec{B}.\nabla$, then integrating by parts gives

$$\delta W_S = -\pi \oint J_\chi d\chi \left(\frac{i}{n} \right) \frac{\vec{B}.\nabla \xi_\psi^*}{B^2} \left[R^2 B_p^2 \frac{i}{n} \frac{\partial}{\partial \psi} \vec{B}.\nabla \xi_\psi + \vec{B}.\vec{J} \xi_\psi \right] \tag{15}$$

Finally, using $\vec{B}.\nabla \xi_\psi = \nabla \psi.\vec{B}_1$ and adding δW_V we obtain

$$\delta W_S + \delta W_V = \pi \oint J_\chi d\chi \left(\frac{i}{n} \right) \frac{\left(\nabla \psi.\vec{B}_1^* \right)}{B^2} \left\{ R^2 B_p^2 \left[\left| \frac{i}{n} \frac{\partial}{\partial \psi} \left(\nabla \psi.\vec{B}_1 \right) \right| \right] - \vec{B}.\vec{J} \xi_\psi \right\} \tag{16}$$

The term in $\{\}$ is exactly Eq. 9 with $\vec{\sigma} = \vec{J}_0 \xi_\psi$ as in Eq. 8. Therefore marginal stability of the Peeling mode corresponds (at high-n) to neglecting δW_F, and solving $\delta W_S + \delta W_V = 0$. This suggests we should define the high-n Peeling mode as a mode with $\delta W_F \ll \delta W_S + \delta W_V$, and whose subsequent stability is determined by δW_S and δW_V.

X-point Plasmas: We start from Eq. 16, using $\nabla\psi.\vec{B}_1^V = \nabla\psi.\vec{B}_1 = \vec{B}.\nabla\xi_\psi$, to write

$$\delta W_S + \delta W_V = \pi \oint J_\chi d\chi \left(\frac{i}{n}\right) \frac{\vec{B}.\nabla\xi_\psi^*}{B^2} \left\{ R^2 B_p^2 \left[\left| \frac{\frac{\partial}{\partial\psi}\nabla\psi.\vec{B}_1}{\nabla\psi\vec{B}_1} \right|\right] \left(\frac{i}{n}\right) \vec{B}.\nabla\xi_\psi - \vec{J}.\vec{B}\xi_\psi \right\} \quad (17)$$

Now we consider the stability of plasmas to a single Fourier mode $\xi = \xi_m(\psi)e^{im\theta - in\phi}$ where $\theta = \frac{im}{q}\int^\chi \nu d\chi'$ is the usual straight field-line poloidal coordinate, finding

$$\delta W = -\pi \left|\xi_m\right|^2 \Delta \left(\Delta\hat{\Delta}' + \hat{J}\right) \quad (18)$$

where $\Delta = \frac{m-nq}{nq}$, $\hat{J} = \oint dl \frac{I}{R^2 B_p} \frac{\vec{J}.\vec{B}}{B^2}$, and $\hat{\Delta}' = \left[\left| \oint dl B_p \frac{I^2}{R^2 B^2} \frac{\frac{\partial}{\partial\psi}\nabla\psi.\vec{B}_1}{\nabla\psi.\vec{B}_1} \right|\right]$ with $dl = J_\chi B_p d\chi$ an element of arc length in the poloidal cross-section. If the plasma is approximated as a vacuum, then it may easily be seen for a circular cross-section that $\hat{\Delta}'$ is a negative constant of order $m \sim nq$. Equation 18 is minimised for $\Delta = -\hat{J}/2\hat{\Delta}'$, giving $\delta W = +\pi \left|\xi_m\right|^2 \hat{J}^2/4\hat{\Delta}'$. Hence the Peeling mode remains unstable for profiles that are arbitrarily close to a separatrix with an X-point, but provided X-point plasmas have $\hat{\Delta}' \sim -nq$ then $\delta W \to 0$ as $q \to \infty$,[2] in agreement with [3, 4, 8]. Current work should confirm whether $\hat{\Delta}' \sim -nq$ for an X-point plasma[9].

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