Statistical Analysis of Neutral Particle Diagnostic Data

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Measurements of kinetic energy distributions of neutral atoms escaping from magnetically confined plasma are used in controlled fusion experiments as a method to investigate the ion component distribution function and its evolution due to the application of various plasma heating schemes. The ion distribution function reflects the kinetic effects, the single particle confinement properties depending on the particular magnetic configuration, the finite β effects such as MHD induced fast ion losses, radial electric field effects, etc. The nuclear fusion reaction rate is determined by the ion distribution and thus its studies at suprathermal energies near the rate coefficient curve maximum are of primary importance.

The probability density function (PDF) $f(E)$ for kinetic energies of neutral H^0 particles escaping from the plasma of a magnetic confinement fusion device in a general form is given by

$$f(E) = A e^{\int_{\rho_{min}}^{1} Q^-(\tilde{\rho})\lambda_{mfp}^{-1}(E,\tilde{\rho})d\tilde{\rho}} \int_{\rho_{min}}^{1} g\left(E,\rho\right)\left[Q^+(\rho)e^{-\int_{\rho_{min}}^{\rho} Q^+(\tilde{\rho})\lambda_{mfp}^{-1}(E,\tilde{\rho})d\tilde{\rho}} - Q^-(\rho)e^{-\int_{\rho_{min}}^{\rho} Q^-(\tilde{\rho})\lambda_{mfp}^{-1}(E,\tilde{\rho})d\tilde{\rho}} \right]d\rho , (1)$$

where A is the normalization constant. The source function for H^0 atoms of energy E within the plasma $g(E,\rho) = n_i(\rho)f_i(E,\rho)\sum_l n^{(l)}(\rho)\langle\sigma v\rangle^{(l)}$ is expressed via the local plasma proton distribution $n_i(\rho)f_i(E,\rho)$ and the sum of rates over all targets for the electron capture process. The derivatives $Q^+(\rho) = d\Lambda/d\rho > 0$ and $Q^-(\rho) = d\Lambda/d\rho < 0$ of the sight line distance Λ along the two intervals between $\rho = 1$ and $\rho = \rho_{min}$ are obtained from the known structure of magnetic surfaces $\rho = const$. The neutral flux attenuation enters in the form of Poisson exponents, where $\lambda_{mfp}(E,\rho)$ is the H^0 mean free path with respect to all electron loss reactions.

Ideally, the passive diagnostic data is an array $(E_1, \ldots, E_N)$ of energies of escaped neutral particles measured along a certain observation direction, and N is the total number of particles collected during a certain time interval. This array is a sample of realizations of the random variable E distributed according to the law (1). Such form of data is achievable with solid state detectors by using pulse height analysis techniques, while the other analyzers, e.g. $\vec{E} \parallel \vec{B}$ ones, intrinsically form a histogram of the incoming particle energies over a certain number of subintervals called energy channels. Technical details may be found in [1, 2]. The formulation of the problem considered here is to obtain an estimate $f^{(*)}(E)$ of the unknown exact probability density function $f(E)$ of neutral particle energies from the experimental data. The sought function preferably should satisfy a specified precision criterion. The obtained PDF estimate is then to be used to reconstruct the ion distribution for further analysis.

Assuming a predefined theoretical PDF $f(E)$ one can carry out a numerical experiment by generating a sample of escaped atom energies for given plasma parameters and experimental conditions. We apply the inverse cumulative distribution function (CDF) approach. First, a sample of pseudorandom numbers $(u_1, \ldots, u_N)$ uniformly distributed within the $[0,1)$ interval is generated using an algorithm from [3]. Then, the energy values are calculated as solutions of the equation $F(E_j) = u_j$, where $F(E) = \int_0^E f(\tilde{E})d\tilde{E}$ is the CDF. These simulation results can be supplied as input data for the PDF estimation procedure to test its performance, since the original exact $f(E)$ used in the simulation is known.

Two typical ion energy distribution laws have been used in the numerical simulation, namely, (a) Maxwellian distribution with ion temperature T_i

$$f_i(E) = \frac{2}{\sqrt{\pi}}\frac{1}{T_i}\sqrt{\frac{E}{T_i}}\exp(-E/T_i); \quad F_i(E) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{E}{T_i}\right), \tag{2}$$

where $\gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt$ is the lower incomplete gamma-function, and (b) the classical slowing down distribution for a delta-like fast ion source function $S(v-v_0) = \dfrac{S_0}{4\pi v^2}\dfrac{e^{-\frac{(v-v_0)^2}{\epsilon^2}}}{\epsilon\sqrt{\pi}}$ [4]

$$f_i(v) = \frac{S_0}{8\pi}\frac{\tau_s}{v^3 + v_c^3}\left(erf\left(\frac{v^*(v,t)-v_0}{\epsilon}\right) - erf\left(\frac{v-v_0}{\epsilon}\right)\right), \tag{3}$$

where the slowing down time $\tau_s = \dfrac{3m_p T_e^{3/2}}{4\sqrt{2\pi}n_e e^4 \Lambda m_e^{1/2}}$, the critical velocity $v_c^3 = \dfrac{3\sqrt{2\pi}T_e^{3/2}}{2m_p m_e^{1/2}}$, Λ is the Coulomb logarithm, and $v^*(v,t) = \left((v^3+v_c^3)e^{3t/\tau_s} - v_c^3\right)^{1/3}$. The ion velocity $v = \sqrt{2E/m_p}$, v_0 is the injection velocity, S_0 and ϵ determine the source rate and width, and t is the time. Fig. 1 (a) shows the Maxwellian PDF for two different T_i values and Fig 1 (b) shows the classical slowing down PDF at t = 0.8 s for injection energy E_0 = 150 keV and two different pairs of the target plasma n_e and T_e values. Histograms of the corresponding pseudorandom number samples governed by these PDFs are shown in Fig. 1 (c) and (d).

As an improvement of the neutral particle diagnostic data analysis, we have applied the empirical probability density estimation [5] and kernel smoothing techniques, e.g., [7-9]. The empirical PDF

$$f^{(e)}(E) = \frac{F^{(e)}(E+\lambda) - F^{(e)}(E-\lambda)}{2\lambda}, \quad \lambda > 0, \tag{4}$$

is in fact a central difference derivative of the empirical CDF $F^{(e)}(E) = \dfrac{\#\{E_j : E_j \le E\}}{N}$. To describe the accuracy, Kolmogorov statistic $D_N = \sup_E\left|F^{(e)}(E) - F(E)\right|$ is used and its asymptotic distribution $\lim_{N\to\infty}\mathbf{P}\left(D_N\sqrt{N} \le y\right) = K(y)$, $y > 0$ [6]. As shown in [5], if $\left|f'(E)\right| \le C$ $\forall E \in [0,+\infty)$, then $\left|f^{(e)}(E) - f(E)\right| \le \dfrac{y/\sqrt{N}}{\lambda} + \dfrac{C\lambda}{2}$. The optimum value of the
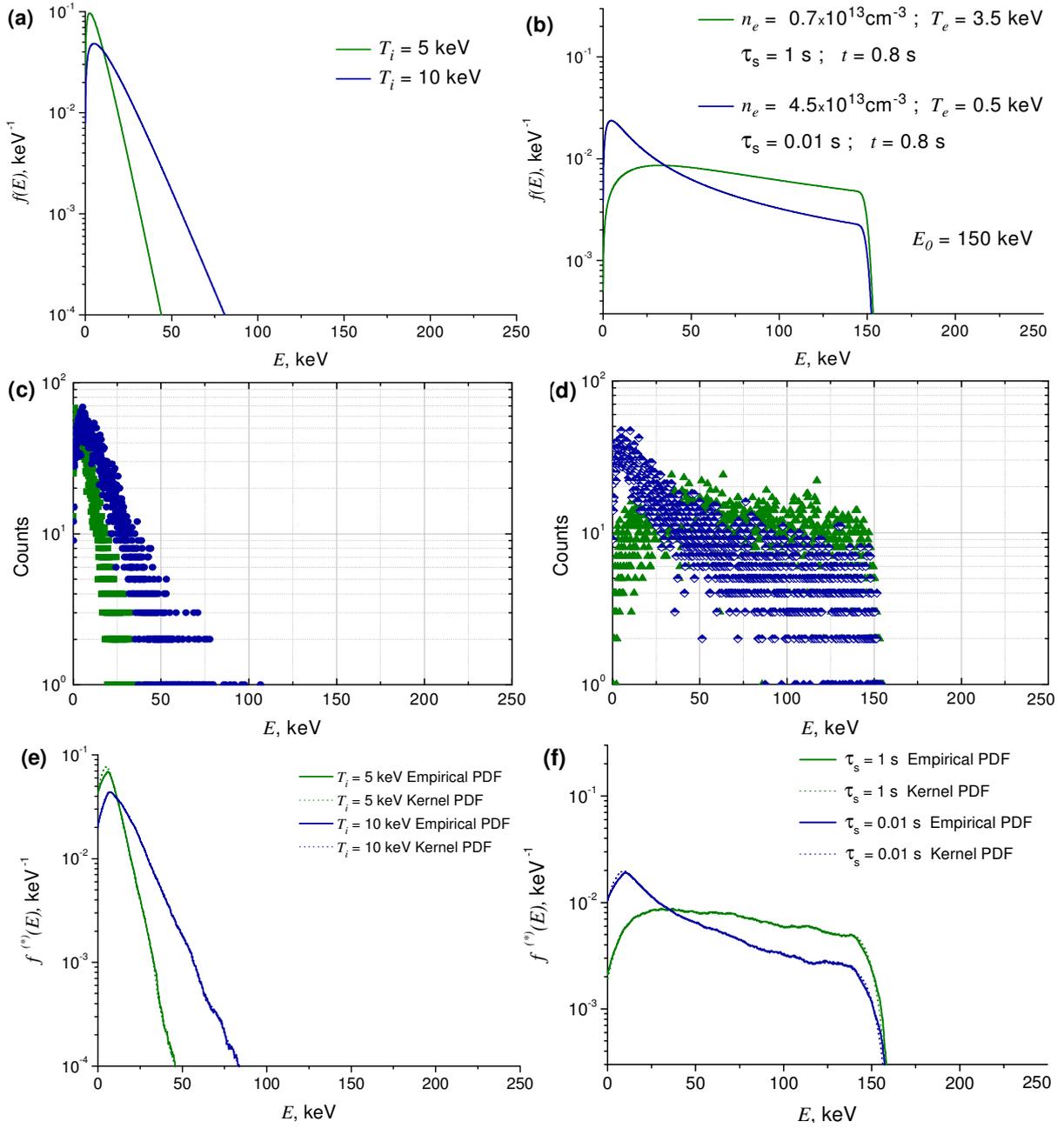
Fig. 1. (a) Maxwellian PDF for $T_i = 5$ keV (green) and $T_i = 10$ keV (blue); (b) classical slowing down PDF for $\tau_s = 1$ s (green) and $\tau_s = 0.01$ s (blue); (c) histograms of pseudorandom number samples distributed according to the laws shown in (a); (d) histograms of pseudorandom number samples distributed according to the laws shown in (b); (e) empirical (solid lines) and kernel (dotted lines) PDFs calculated from Maxwellian law pseudorandom number samples shown in (c); (f) empirical (solid lines) and kernel (dotted lines) PDFs calculated from slowing down distribution law pseudorandom number samples shown in (d).

parameter in (4) is then $\lambda^* = \sqrt{\dfrac{2y}{C}} N^{-1/4}$. The other method is to use the kernel PDF estimate

$$f^{(k)}(E) \;=\; \frac{1}{Nh}\sum_{j=1}^{N} K\!\left(\frac{E - E_j}{h}\right), \quad h > 0 \tag{5}$$

determined by the kernel function $K(z)$ and the kernel bandwidth h. The optimum kernel derived in [8] is $K(z) = \dfrac{3}{4}\left(1 - z^2\right) I_{(-1,\,1)}(z)$, where $I_{(-1,\,1)}(z)$ equals unity within (-1, 1) and equals nought outside. The performance criterion of this method is the mean integrated
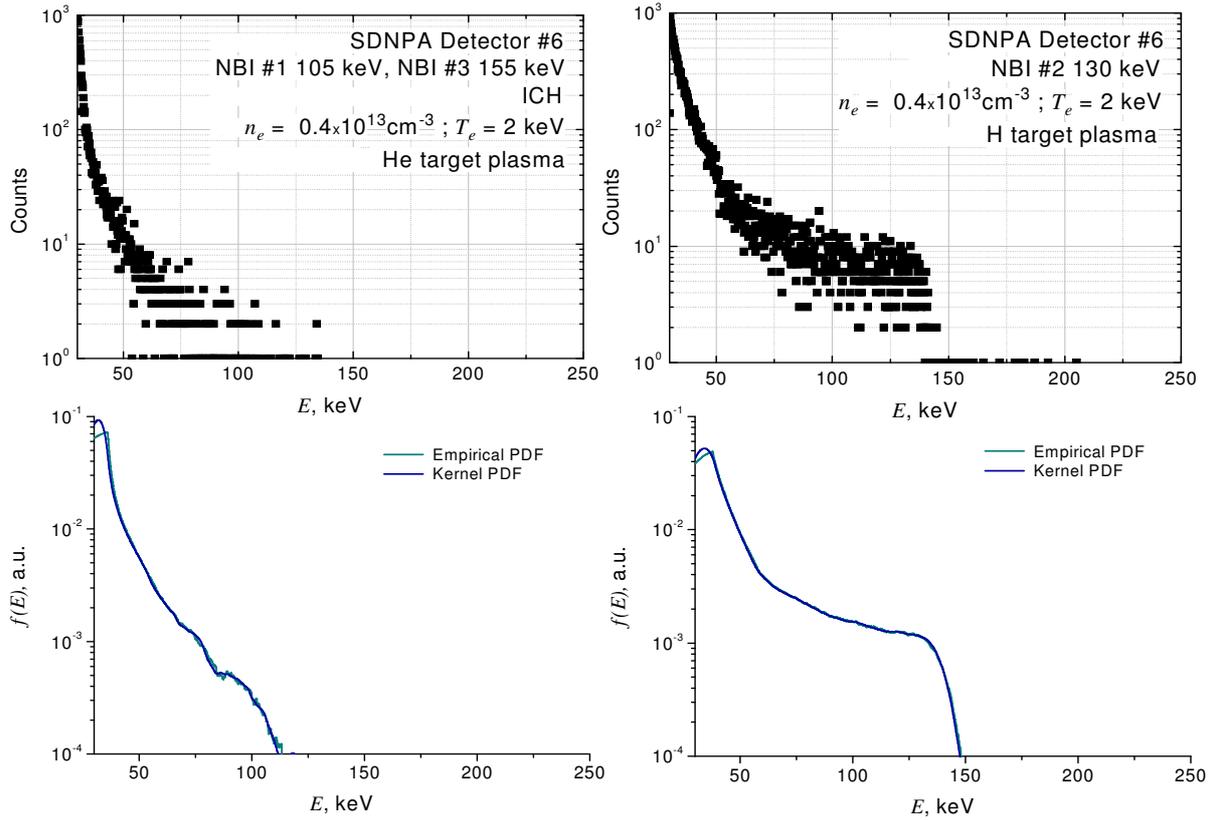
Fig. 2. Experimental H^0 energy spectra (upper) and PDF estimates (lower) for two different ion heating schemes.

squared error $MISE\left(f^{(k)}, f\right) = \left\langle \int_0^{+\infty} \left[f^{(k)}(E) - f(E)\right]^2 dE \right\rangle$ and its "asymptote" for $N \gg 1$

$$AMISE\left(f^{(k)}, f\right) = \frac{c_1}{Nh} + \frac{c_2 c_3^2 h^4}{4} , \text{ where } c_1 = \int K^2(z)dz \text{ ; } c_2 = \int \left(f''(z)\right)^2 dz \text{ ; } c_3 = \int z^2 K(z)dz .$$

The optimum bandwidth minimizing AMISE is then $h^* = c_1^{1/5} c_2^{-1/5} c_3^{-2/5} N^{1/5}$ as shown in [7].

These methods have been tested by reconstructing the probability density function from the generated pseudorandom number samples assuming Maxwellian and classical slowing down plasma ion energy distribution shapes for different temperatures and different slowing down times. The test results are shown in Fig. 1 (e) and (f). The analysis of passive chord-integrated experimental data obtained with the Silicon Detector Neutral Particle Analyzer [10] on Large Helical Device is illustrated in Fig. 2. Similar results have been obtained with empirical PDF and kernel smoothing techniques. Both methods require a certain choice of the bandwidth parameter. An automatic choice is preferable for routine data processing.

References

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