Anomalous particle and heat transport modeled by the combined random walk in position and momentum space

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Abstract

We introduce the combined Continuous Time Random Walk (CTRW) in position and momentum space, in the form of two coupled integral equations that are solved numerically. An application is made to a toroidally confined plasma that undergoes off-center injection of cold plasma (off-axis fueling), focusing on phenomena of anomalous transport.

Introduction

It has been shown that Random Walk in the form of Continuous Time Random Walk (CTRW; e.g. [1], first applied to plasma in [2]) can successfully model observed phenomena of anomalous transport in confined plasmas, such as profile stiffness, uphill transport in response to off-axis fueling (e.g. [3], [4], the so-called critical gradient model).

So-far, all applications of CTRW were done in position space only, the evolution of the velocity was not included. Our aim is to extend the CTRW to include in parallel, besides position space, also momentum space (combined CTRW: C-CTRW), which is a more realistic, dynamic model. As in the cited works, our intended application is to confined plasma.

The combined CTRW in position and momentum space

Position, time, and momentum are considered, and we focus on individual particles. The particles evolve by performing random steps Δx_i in position space, Δp_i in momentum space, and Δt_i in time. Formally, we assume the distribution of increments $q_3(\Delta x, \Delta p, \Delta t)$, i.e. the probability to make a jump Δx and to spend a time Δt in the jump and to change the momentum by Δp, to factorize,

$$q_3(\Delta x, \Delta p, \Delta t) = \delta(\Delta t - |\Delta x|/|v|)q_x(\Delta x)q_p(\Delta p),$$

where Δx and Δp are independent, and Δt is given as $\Delta t = |\Delta x|/|v|$, i.e. Δt is the time a particle spends in a free flight, where the velocity v is a function of the instantaneous momentum p and is variable.

The C-CTRW equations are derived from the CTRW equations of e.g. [5] in integral form by carefully adding the momentum. The final equations of the C-CTRW are as follows: The probability $P(x,p,t)$ to be at x and have a momentum p at time t is given as

$$P(x,p,t) = \int dp' \int dx' \int_0^t dt'\, Q(x',p',t')\, \Phi(x-x',t-t';v(p))\, q_p(p-p'),$$

where the auxiliary function Q is the distribution of turning points, and it is determined as

$$\begin{aligned} Q(x,p,t) \;&=\; \int dp' \int dx' \int_0^t dt'\, Q(x',p',t')\, \delta(t-t'-|x-x'|/v(p)) \\ &\quad \times q_x(x-x')\, q_p(p-p') + \delta(t)P(x,p,0) + S(x,p,t), \end{aligned}$$

with $S(x,p,t)$ a source term and $P(x,p,0)$ the initial condition. Finally, Φ is the probability to make a jump of length Δx or longer and of duration Δt or longer, and to be at time t at position x, $\Phi(\Delta x,\Delta t,v) = \frac{1}{2}\,\delta(|\Delta x|-v\Delta t) \int_{|\Delta x|}^{\infty} dx' \int_{\Delta t}^{\infty} dt'\, \delta(t'-|x|'/v)\, q_x(x')$. Note that Φ is given, it is derived from the known $q_x(\Delta x)$.

We solve the C-CTRW equations numerically with a pseudospectral method, where the unknown functions are expanded in terms of Chebyshev polynomials, which allows to achieve high precision with reasonable computing time and memory use.

Application to off-axis fueling

We consider diffusion along the minor radius (x-direction), with p the momentum along x, and the system is finite in the x-direction, The particles are considered to be electrons, drawn from a thermal distribution at injection. The initial distribution is $P(x,p,t=0)=0$. We consider off-axis fueling with (i) a spatially uniform background source (T=8keV), together with (ii) a spatially localized source of colder plasma off the center (T= 3keV), where both sources act uniformly over time.

The kind of diffusion — normal, sub-, or super-diffusion — is determined solely by the choice of q_x and q_p. Of basic interest are two cases, small increments and large increments, respectively, where small means small compared to system size, and large means comparable to or even larger than the system size. For small increments we use Gaussian distributions, $q_x^{(Gauss)}(\Delta x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\Delta x}{2\sigma^2}}$, with σ small, and for large increments, distributions with a power-law at large Δx are used (Levy type distributions), $q_x^{(pl)}(\Delta x) = A\Delta x^{-\alpha}$.

The problem is then how to choose q_x and q_p such that the random walk is relevant and reproduces anomalous transport phenomena in confined plasmas. Widely discussed is the critical gradient model (e.g. [3]), where the distribution of increments depends on the local density gradient dn/dx, the increments are small if the gradient is small, and they are large if the gradient

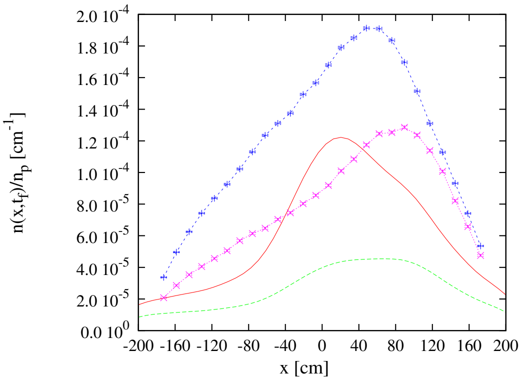

Figure 1: Number density $n(x,t)/n_p$ as a function of x, for the mixed model with Δp Gaussian (solid - red) and power-law (long dashes - green), and for the critical gradient model with Δp Gaussian (short dashes - blue, scaled with a factor $1/3$ for better visualization) and with Δp power-law (dotted - violet).

is large,

$$q_x^{(crit)}(\Delta x) = \begin{cases} q_x^{(Gauss)}(\Delta x), & \text{if } \left|\frac{dn}{dx}\right| < c_{cr} \\ q_x^{(pl)}(\Delta x), & \text{if } \left|\frac{dn}{dx}\right| \geq c_{cr} \end{cases}$$

A second choice we make is the mixed model. Since generally the plasma conditions allow more easily anomalous diffusion towards the edge, we let the distribution of increments be spatially dependent, the increments are small near the center and they are large towards the edge,

$$q_x^{(mixed)}(\Delta x, x) \;=\; f_1(x)\, q_{\Delta x}^{(Gauss)}(\Delta x) \;+\; (1 - f_1(x))\, q_x^{(pl)}(x),$$

with $f_1(x)$ a function that equals one in the center and decays towards the edges.

In the following, we compare results from the mixed model and the critical gradient model, exploring at the same time in both models the two sub-cases of small (Gaussian distributed) and large (power-law distributed) momentum increments. In the following, the off-axis source and the uniform source are of equal strength.

The density profiles in stationary state are shown in Fig. 1. The mixed model shows higher stiffness, but lower density, i.e. confinement is worse. The stiffness is highest with Δp Gaussian distributed. The critical gradient model has higher density, i.e. better confinement, but the peaks are clearly off-center, i.e. it has lower stiffness. In both models, stiffness deteriorates with power-law distributed momentum increments. On the contrary, stiffness in the temperature profiles is achieved only with the Δp power-law distributed.

The dynamic particle flux is determined as $\Gamma(x) = n(x)\langle v(x)\rangle$, with $n(x)$ the density, $< v(x) >$ the local average velocity, and the dynamic heat flux is defined as $q(x) = \frac{1}{2}k_B T(x) n(x)\langle v(x)\rangle$.

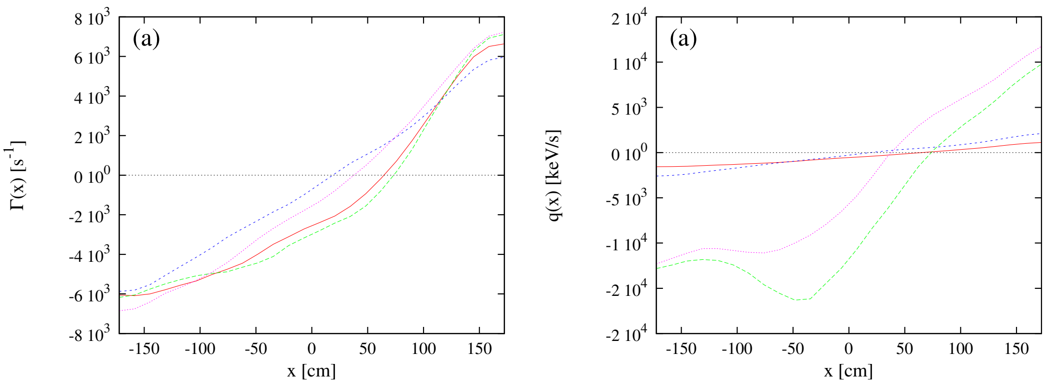

Figure 2: Shown are $\Gamma(x)$ (left panel) and $q(x)$ (right panel) as a function of x, as yielded by the mixed model, for strong off axis loading with Δp Gaussian (solid - red) and power-law (long dashes - green) and for weak off axis loading again with Δp Gaussian (short dashes - blue) and with power-law (dotted - violet)

They are both shown in Fig. 2. Plotting them against the gradients $-\partial_x n(x)$ and $-\partial_x T(x,)$, respectively, we find clear features of non-uniqueness, the fluxes are not single valued functions of the respective gradients, so that the particle and the heat transport is incompatible with the classical approach, mainly due to the non-locality of the transport process.

Conclusion

The combined CTRW in position and momentum space was introduced, in the form of a integral equations, together with a method to solve them numerically. The results show that anomalous transport phenomena can be addressed by the model (including profile stiffness). The particle and the heat flux are both incompatible with the classical approach. Most important, we find that the momentum space dynamics influences the position space dynamics directly, including the nature of anomalous transport phenomena.

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