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Coupling of waves with positive and negative energy as a universal mechanism for MHD instabilities of flowing media

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Energy consideration is of primary significance in stability analysis of different magneto-hydrodynamic (MHD) systems. It is well known that the energy associated with the waves in moving media may change the sign and become negative. For the negative energy wave to be excited the energy should be withdrawn from the system. Therefore, such a wave is a potential source of instability. In a conservative system, the instability can occur due to the simultaneous excitation of positive and negative energy waves. In this case, energy is transferred from the negative energy wave to the positive energy wave, allowing both modes to grow and the total energy to remain constant. Classical examples of such instabilities are Kelvin-Helmholtz instability in neutral fluids and plasma-beam instability in plasmic media.

In this report we calculate the energy of the eigenmodes in ideal MHD and show that the coupling of negative and positive energy waves appears to be a universal mechanism of MHD instabilities in flowing media. Consider linearized dynamics of displacement vector $\boldsymbol{\xi}$ [1]

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}+2\rho(\mathbf{V}\cdot\nabla)\frac{\partial\boldsymbol{\xi}}{\partial t}-\mathbf{F}[\boldsymbol{\xi}]=0, \tag{1}$$

where ρ and $\mathbf{V}$ are stationary values of fluid density and velocity, respectively, $\mathbf{F}[\boldsymbol{\xi}]$ is linearized force operator. Force operator $\mathbf{F}[\boldsymbol{\xi}]$ is Hermitian (self-adjoint) in the following sense,

$$\int\boldsymbol{\eta}\cdot\mathbf{F}[\boldsymbol{\xi}]\,d^3\mathbf{r}=\int\boldsymbol{\xi}\cdot\mathbf{F}[\boldsymbol{\eta}]\,d^3\mathbf{r}, \tag{2}$$

while the second term in Eq. (1) is antisymmetric:

$$\int\rho\boldsymbol{\eta}\cdot(\mathbf{V}\cdot\nabla)\boldsymbol{\xi}\,d^3\mathbf{r}=-\int\rho\boldsymbol{\xi}\cdot(\mathbf{V}\cdot\nabla)\boldsymbol{\eta}\,d^3\mathbf{r}. \tag{3}$$

Integration in Eqs. (2) and (3) is performed over the fluid volume under the assumption that displacements on the boundary vanish.

Assuming vector $\boldsymbol{\xi}$ to be complex, we obtain the expression for energy of perturbations

$$E=\frac{1}{2}\int\left(\rho\left|\frac{\partial\boldsymbol{\xi}}{\partial t}\right|^2-\boldsymbol{\xi}^*\cdot\mathbf{F}[\boldsymbol{\xi}]\right)d^3\mathbf{r}, \tag{4}$$

where star denotes complex conjugation. The energy is conserved, i.e., $\partial E/\partial t=0$.

We look for a normal-mode solutions to Eq. (1) in the form

$$\boldsymbol{\xi}(\mathbf{r},t) = \hat{\boldsymbol{\xi}}(\mathbf{r})e^{-i\omega t}. \tag{5}$$

Then, the equation of motion (1) leads to eigen-value problem

$$\omega^2\rho\hat{\boldsymbol{\xi}} + 2i\omega\rho(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}} + \mathbf{F}[\hat{\boldsymbol{\xi}}] = 0. \tag{6}$$

Multiplying this equation by complex conjugate $\hat{\boldsymbol{\xi}}^*$ and integrating over the fluid volume, we arrive at quadratic equation for eigen-frequency ω,

$$A\,\omega^2 - 2B\,\omega - C = 0, \tag{7}$$

$$A = \int \rho|\hat{\boldsymbol{\xi}}|^2 d^3\mathbf{r} > 0, \quad B = -i\int \rho\hat{\boldsymbol{\xi}}^*\cdot(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}}\, d^3\mathbf{r}, \quad C = -\int \hat{\boldsymbol{\xi}}^*\cdot\mathbf{F}[\hat{\boldsymbol{\xi}}]\, d^3\mathbf{r}.$$

Solving Eq. (7) we find

$$\omega_{1,2} = \frac{B \pm \sqrt{B^2+AC}}{A}. \tag{8}$$

Here and subsequently, subscript "1" corresponds to plus, while subscript "2" is combined with minus. Note that all coefficient in Eq. (7) are real. Hence, the instability in the system is possible if and only if $B^2 + AC < 0$ for some eigenmode.

The energy of the eigen-mode with eigen-frequency (8) is

$$E = \frac{1}{2}(A\,|\,\omega|^2 + C)e^{2\gamma t}, \tag{9}$$

where $\gamma = \mathrm{Im}\ \omega$ is imaginary part of eigen-frequency. For unstable mode, $B^2 + AC < 0$, so $|\,\omega_{1,2}|^2 = -C/A$, and the energy is

$$E_{1,2} = 0. \tag{10}$$

For stable mode, $B^2 + AC \geq 0$, so $\gamma = 0$ and the energy is

$$E_{1,2} = \frac{\sqrt{B^2+AC}}{A}\left(\sqrt{B^2+AC} \pm B\right) = \pm\,\omega_{1,2}\sqrt{B^2+AC}. \tag{11}$$

Hence, energy of each stable eigen-mode (1 and 2) changes the sign if and only if the frequency of this mode changes the sign. We note that all negative energy waves are non-symmetric eigen-modes, i.e., they have spatial dependence along the equilibrium flow, so the coefficient $B \neq 0$. For symmetric modes or in the absence of flow, we have $B = 0$ and the energy is

$$E = \frac{1}{2}(|C| + C).$$

Therefore, energy of symmetric modes is never negative, and their stability can be investigated by use of standard energy principle [1]. In a case of non-symmetric modes, the energy principle fails and modified approach should be used (see, e.g., [2]).

In order to verify the above analytical results, we calculate the energy of eigenmodes of incompressible fluid rotating in homogenous transverse magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. The equilibrium velocity profile corresponds to the electrically driven flow in circular channel and has a form

$$\mathbf{V} = r\Omega(r)\mathbf{e}_\varphi, \quad \Omega(r) = \frac{\Omega_1 r_1^2}{r^2} \tag{12}$$

in cylindrical system of coordinates $\{r,\varphi,z\}$. Here, r_1 and r_2 are inner and outer radii of the channel, respectively, and Ω_1 is the angular velocity at r_1. This type of flow profile is used in new experimental device [3] for laboratory testing of magnetorotational instability (MRI).

A detailed eigenmode analysis of such flow has been performed in Ref. [4]. Assuming

$$\boldsymbol{\xi}(\mathbf{r},t) = \boldsymbol{\xi}(r)e^{-i\omega t+im\varphi+ik_z z}$$

one obtains an expression for energy of stable eigenmode with frequency ω [5]:

$$E = \pi\rho h\omega \int_{r_1}^{r_2}\left[\bar{\omega}\xi_r^2 + \frac{\bar{\omega}}{m^2+k_z^2r^2}\left(\frac{\partial(r\xi_r)}{\partial r}\right)^2 + \frac{4\bar{\omega}k_z^2r^2\Omega^2\omega_A^2\xi_r^2}{(\omega_A^2-\bar{\omega}^2)^2(m^2+k_z^2r^2)} + \frac{2m\Omega\xi_r}{m^2+k_z^2r^2}\frac{\partial(r\xi_r)}{\partial r}\right]rdr,$$

where h is the height of the channel, $\omega_A = k_z B_0/\sqrt{4\pi\rho}$ is Alfven frequency and $\bar{\omega} = \omega - m\Omega$ is Doppler "shifted" eigen-frequency. For axisymmetric eigen-modes with $m = 0$ we have

$$E = \pi\rho h\omega^2 \int_{r_1}^{r_2}\left[\xi_r^2 + \frac{1}{k_z^2r^2}\left(\frac{\partial(r\xi_r)}{\partial r}\right)^2 + \frac{4\Omega^2\omega_A^2\xi_r^2}{(\omega_A^2-\omega^2)^2}\right]r\,dr.$$

Therefore, the energy of axisymmetric eigen-modes is always positive if $\omega \neq 0$. Formally, this case is described by Eq. (11) with coefficient $B = 0$.

In Fig. 1, the calculated dependencies of frequency and energy for two potentially unstable eigen-modes on the value of the parameter Ω_1/ω_A are shown. In the axisymmetric case ($m = 0$), both branches of energy are

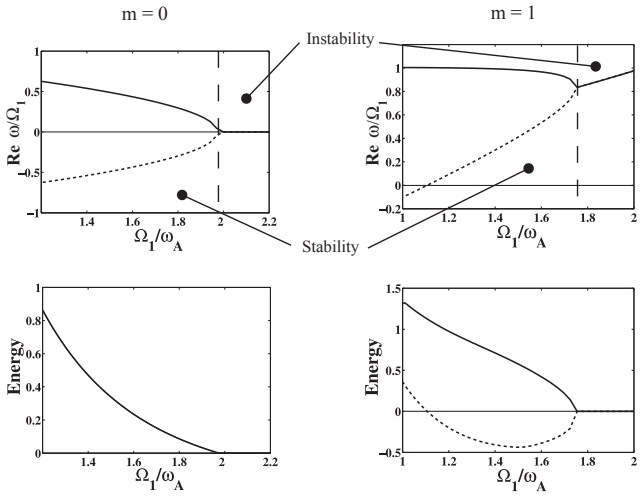

Figure 1: Dependence of eigen-frequency and energy on ratio Ω_1/ω_A for two most unstable eigen-modes with azimuthal numbers $m = 0$ and $m = 1$. Energy is given in arbitrary units.

positive and coincident. The merging point of eigen-frequencies corresponds to $\Omega_1/\omega_A \approx 2.0$ which is the threshold of magnetorotational instability for $m = 0$. The nature of axisymmetric MRI is not related to the subject of negative energy waves and can be explained by the mechanism similar to one of Rayleigh-Taylor instability [6].

For $m = 1$ the MRI threshold is $\Omega_1/\omega_A \approx 1.7$. When $1.1 < \Omega_1/\omega_A < 1.7$ the positive and negative energy waves can coexist in the system. At $\Omega_1/\omega_A \approx 1.1$ the frequency ω_2 changes the sign, so both energy branches become positive. This result suggests that the non-symmetric instability in the ideal MHD system with flow is associated with coupling of positive and negative energy waves. Such coupling occurs when the frequencies of both modes coincide. This resonance condition allows energy to be transferred from the negative energy wave to the positive energy wave leading to instability.

It should be noted that merging points in Fig. 1 determine the magnetorotational instability threshold. In the flow given by Eq. (12) this threshold decreases with azimuthal number m, as discussed in Ref. [4]. For large m it approaches the asymptote

$$\frac{\Omega_1}{\omega_A} = \frac{2}{m(1-r_1^2/r_2^2)}. \qquad (13)$$

The calculated dependence of MRI threshold on m is presented in Fig. 2.

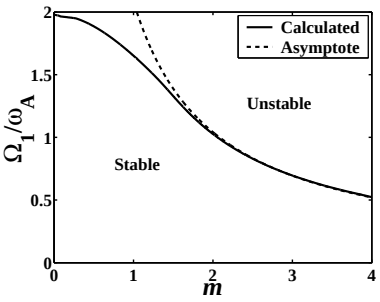

Figure 2: Dependence of MRI threshold on azimuthal mode number m (solid line) and its asymptote (dashed line).

In conclusion, we have shown the principal difference between symmetric and non-symmetric modes. Our results suggest that MHD instabilities of non-symmetric modes in ideal fluids with flows can be explained as a coupling of originally stable positive and negative energy waves.

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