

Three-Dimensional Magnetic Field Calculation for a Localised Current Distribution in Unbounded Space

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Algorithm for field Calculation

The topic of this work is a new method for the rapid numerical field calculation from a current with arbitrary spatial variation. It is designed for applications in toroidal fusion machines, but extensible to other devices. The displacement current is considered to be negligible. The evaluation of the 6 dimensional Biot-Savart integral can lead to excessive computing time for this type of problem. We instead solve Ampère's equation for the magnetic vector potential. A cylindrical coordinate system (R, φ, Z) aligned with a suitably defined vertical axis is introduced. Since we do not deal with radiation phenomena, the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ is appropriate and also serves for simplifying the equations. Ampère's equation $-\Delta \mathbf{A} = \mu_0 \mathbf{J}$ is then written in cylindrical coordinates, and decomposed in toroidal Fourier modes along the φ coordinates. With the definition $\Psi = RA_R$, we arrive at

$$-\Delta_n A_{Zn} = \mu_0 J_{Zn} \quad (1)$$

$$-\Delta_n \Psi_n = \mu_0 R J_{Rn} + 2 \frac{\partial A_{Zn}}{\partial Z} \quad (2)$$

$$\Delta_1 A_{\varphi 0} = \mu_0 R J_{\varphi 0}, \quad \text{for } n = 0 \quad (3)$$

$$-A_{\varphi n} = \frac{i}{n} \left(\frac{\partial \Psi_n}{\partial R} + \frac{\partial A_{Zn}}{\partial Z} \right), \quad \text{for } n \neq 0 \quad (4)$$

$$\text{with} \quad \Delta_n := \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} - \frac{n^2}{R^2} \quad (5)$$

Ampère's equation has now been decoupled into a set of 3 two-dimensional equations equations that are solved for each mode in the order (1), (2), (3) resp. (4). Rapid standard methods are used to invert the Laplace operator (e.g. [2]). These rapid field solvers require a finite computing grid enclosing the current distribution and therefore boundary conditions. Boundary conditions at infinity are mapped to the finite grid with a technique developed by Lackner [1]. This method is applied to equations (1) and (3). It has to be modified for equation (2), because the term containing A_Z is unequal zero everywhere. We split Ψ_n into a contribution $\Psi_{n,J}$, and the contribution from $\partial A_{nZ}/\partial Z$. Combined with (1), one arrives at the bi-harmonic equation

$$(\Delta_n)^2 \Psi_{n,A} = 2\mu_0 \frac{\partial J_{Zn}}{\partial Z} \quad (6)$$

In a first step, equation (6) is solved with boundary $\Psi_{n,A} = 0$ and $\Delta\Psi_{n,A} = 0$ on the computing grid. The resulting functions $\Psi_{n,0}$ and $\Delta\Psi_{n,0}$ are then used to obtain the boundary condition with a corresponding Green's theorem (see [6])

$$\Psi_b = \oint \left(G_n \frac{\partial \Psi_{A,n}^0}{\partial n} + G_n^B \frac{\partial^2 A_{Z,n}}{\partial n \partial Z} \right) dS' \quad (7)$$

This generates the correct boundary condition on the grid for Ψ_{nA} . The Green's functions for Δ_n resp. $(\Delta_n)^2$ are given by [4, 5]

$$G_n(R, R', z) = \frac{1}{\pi \sqrt{RR'}} Q_{n-1/2}(\chi), \quad \chi = \frac{R^2 + R'^2 + z^2}{2RR'} \quad (8)$$

$$G_n^B(R, R', z) = \frac{\sqrt{\rho \left(\frac{\rho}{RR'} + 4 \right)}}{4\pi \left(n^2 - \frac{1}{4} \right)} Q_{n-\frac{1}{2}}^1(\chi), \quad \rho = 2RR'(\chi - 1) \quad (9)$$

where $z := Z - Z'$, and $Q_{n-\frac{1}{2}}^{0,1}(\xi)$ [4] are the associated Legendre function of half-integral degree. These functions can be calculated with effective library routines [7].

Implementation and tests of code amp3d

The discretisation scheme can be chosen freely, depending on the precision requirement. We present an implementation with finite differences and a rapid elliptic solver using the fast Fourier transformation (see [2]), with the project name amp3d. The code is written in standard Fortran 95, and uses the libraries LAPACK (<http://www.netlib.org/lapack>) and FFTW (<http://www.fftw.org>). The field calculation with 600 grid points in radial and 400 points in vertical direction takes less than 0.6 sec per Fourier mode on a 64 bit PC type workstation. The code amp3d was tested on the vacuum ripple field of a tokamak with circular coils, where a semi-analytic solution is known. Figure 1 shows a plot of the toroidal ripple field and demonstrates very good agreement with the numerical calculation. This test is demanding, because finite difference codes are usually not well suited to model sharp gradients of material properties, as the interface between vacuum and a field coil. Higher precision is expected in this case when using finite element codes.

A practical application is the calculation of the vacuum ripple field of a tokamak with D-shaped coils, as the ITER device, shown in figure 2. The effect of the proposed ferromagnetic inserts [10] is not taken into account.

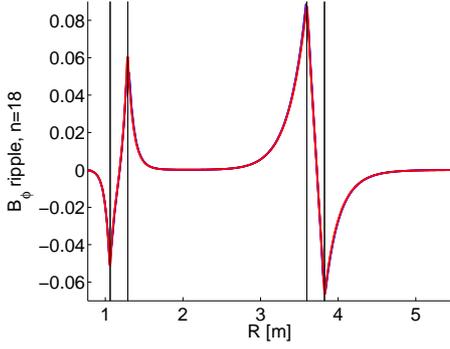


Figure 1: Tokamak with 18 circular coils (Tore Supra). Toroidal ripple field \mathbf{B}_φ on midplane ($Z = 0$) as function of major radius. Deviation: Well below 1 percent except at the coil surface, denoted by the vertical bars.

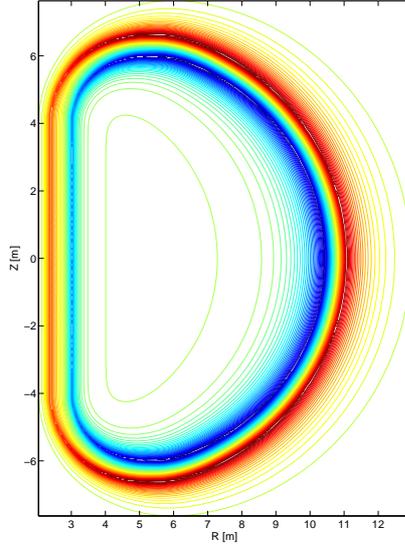


Figure 2: Contour plot of the toroidal ripple field \mathbf{B}_φ of the ITER device with 18 toroidal coils, using coordinates from [9]. The plot shows the ripple field inside the toroidal coil, the plasma region and also the stray field outside the device.

Plasma response to vacuum ripple

During a discharge, the ripple field causes the predominantly axisymmetric equilibrium to develop a toroidal variation of the plasma. Tokamak discharges have a dominating toroidal field component B_T , which allows to expand the force balance $\nabla p = \mathbf{J} \times \mathbf{B}$ in the small parameter $\varepsilon = O(B_p/B_T)$ [8]. The first order term of this expansion gives a correction term for the poloidal current density

$$\tilde{\mathbf{J}}_p^{(1)} = \frac{J_T^{(0)}}{B_T^{(0)}} \tilde{\mathbf{B}}_p^{(0)}, \quad (10)$$

where $\tilde{\mathbf{B}}_p^{(0)}$ is the poloidal vacuum ripple field. The toroidal current density is obtained by $\nabla \cdot \tilde{\mathbf{J}} = 0$, which allows to calculate the resulting field of the plasma response with amp3d. Figure 3 compares the plasma response with the correction of the poloidal flux coming from the vacuum ripple field alone. The poloidal flux function describing the vacuum contribution is formally derived from the local flux function $\psi(R, \varphi, Z)$ with $\nabla \psi \cdot \mathbf{B} = 0$. Expansion of ψ and \mathbf{B} with respect to the vacuum ripple field gives the perturbed poloidal flux as $\tilde{\psi} = -\int R \nabla \Psi_0 \cdot \tilde{\mathbf{B}}_p^{(0)} d\varphi$.

The plasma response is much smaller at the plasma boundary, but it has opposite sign and

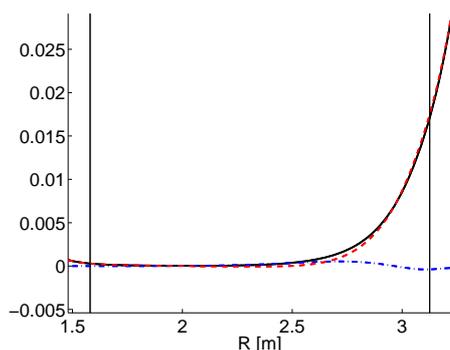


Figure 3: Poloidal flux of ripple correction (Tore Supra discharge 33219) on mid-plane ($Z=0$), as a function of major radius (solid line). Dashed curve denotes vacuum ripple field, dash-dotted plasma response.

partially compensates for the vacuum ripple field. On the other hand, the plasma response contributes to the Shafranov shift in the centre of the plasma. Both results agree with calculations with VMEC for a similar case [11]. The plasma response partially compensates for the vacuum ripple perturbation.

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