Two-dimensional compressional Alfvén eigenmode structure

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Recent experiments in the Mega Ampere Spherical Tokamak have lead to renewed interest in compressional Alfvén eigenmodes (CAEs) [1, 2]. Polarisation measurements have indicated that the perturbed magnetic field is mainly parallel to the equilibrium field, which points to the possibility that the observed modes are CAEs. Growth calculations and a simplified 1D eigenmode analysis are presented in another contribution to this conference [3].

The two-dimensional structure of CAEs has previously been studied both analytically and numerically. The analytical works suffer from several limiting assumptions that are necessary in order to make the problem tractable [4, 5]. The numerical approaches have either not been suitable for large ellipticity tight aspect ratio plasmas [6], or do not include the Hall term [7]. In the present work, the 2D structure is studied by solving the linearised cold plasma Hall-MHD equations numerically. The experimentally observed frequency is typically around half the ion cyclotron frequency, so the Hall term needs to be included. A low β assumption is made, in order to exclude the slow magnetoacoustic mode and to be able to use the cold plasma model. For a spherical tokamak it is also important to treat the variation of the equilibrium magnetic field correctly. Moreover, the parallel wavenumber $k_\parallel$ is allowed to be non-zero, but assumed to be small enough ($k_\parallel^2 \ll \omega^2/v_A^2$) to avoid coupling to the shear Alfvén branch.

The linearised momentum balance equation

$$-i\omega\rho\mathbf{v} = \mathbf{J}_1 \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}_1, \tag{1}$$

the linearised Ohms law

$$\mathbf{E} = \left(\frac{\mathbf{J}_1}{n_e e} - \mathbf{v}\right) \times \mathbf{B}_0 + \frac{\mathbf{J}_0}{n_e e} \times \mathbf{B}_1, \tag{2}$$

and the Maxwell equations are combined to give an equation for the perturbed magnetic field

$$\omega^2\mathbf{B}_1 = \nabla \times \left\{ \left[iF\nabla \times \mathbf{B}_1 - G(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 - G(\nabla \times \mathbf{B}_0) \times \mathbf{B}_1\right] \times \mathbf{B}_0 + iF(\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 \right\}, \tag{3}$$

where $F = -\omega v_A^2/(\Omega_i B_0)$ and $G = v_A^2/B_0^2$ are flux functions and Ω_i is the ion cyclotron frequency. The frequency ω is assumed to be comparable to but lower than Ω_i. The perturbed magnetic field $\mathbf{B}_1$ is represented as the sum of three orthogonal components,

$$\mathbf{B}_1 = b_\parallel \mathbf{B}_0 + B_r \nabla r + b_\wedge \mathbf{T}, \tag{4}$$

where r is a flux label and $\mathbf{T} = \mathbf{B}_0 \times \nabla r$. The three functions $b_\parallel$, B_r, and $b_\wedge$ are governed by the three components of Eq. (3) along $\mathbf{B}_0$, ∇r and $\mathbf{T}$ together with appropriate boundary conditions at the edge of the plasma. In this work, a perfect conductor boundary condition is used, i.e. $B_r = 0$ and $\mathbf{E} \times \nabla r = 0$ at $r = a$. The equations can be simplified by the small $k_\parallel$ assumption, which gives an ordering

$$\omega^2 X_{\text{perturb}} \gg \frac{v_A^2}{B_0^2} (\mathbf{B}_0 \cdot \nabla)^2 X_{\text{perturb}}, \tag{5}$$

where X is a perturbed quantity. Furthermore, a single toroidal mode number n is considered, so that X varies as $\exp[i(-n\varphi - \omega t)]$.

Since F contains ω, the three equations (3) cannot be solved as an eigenvalue problem for ω^2 straight away. Instead, the terms including F in the equation are refined iteratively. First, a starting guess ω_{guess} is made, and the corresponding F_{guess} is calculated. Next, the eigenvalue problem is solved numerically to obtain ω, which is then used as a new ω_{guess}, and the process is repeated. In most cases it converges in only a few steps.

A simple approach to solving the system of equations (3) is to neglect the equilibrium current [the $\nabla \times \mathbf{B}_0$ terms in Eq. (3)]. This corresponds to the method used in previous works, where the conductivity tensor for homogeneous equilibrium magnetic field was used, instead of the full conductivity tensor operator [8]. The three components of Eq. (3) become

$$\omega^2 B_r = |\nabla r|^{-2} \mathbf{B}_0 \cdot \nabla \left[iF \mathbf{T} \cdot \nabla b_\parallel + G B_0^2 \nabla r \cdot \nabla b_\parallel \right], \tag{6}$$

$$\omega^2 b_\wedge = |\nabla r|^{-2} B_0^{-2} \mathbf{B}_0 \cdot \nabla \left[iF - B_0^2 \nabla r \cdot \nabla b_\parallel + G B_0^2 \mathbf{T} \cdot \nabla b_\parallel \right], \tag{7}$$

$$\omega^2 b_\parallel = B_0^{-2} \nabla \cdot \left\{ iF \left[B_0^2 \mathbf{B}_0 \times \nabla b_\parallel - \mathbf{T} (\mathbf{B}_0 \cdot \nabla) B_r + \nabla r (\mathbf{B}_0 \cdot \nabla) b_\wedge \right] + \right.$$
$$\left. + G B_0^2 \left[-B_0^2 \nabla_\perp b_\wedge + \nabla r (\mathbf{B}_0 \cdot \nabla) B_r + \mathbf{T} (\mathbf{B}_0 \cdot \nabla) b_\wedge \right] \right\}. \tag{8}$$

A finite difference numerical scheme has been implemented to solve these eigenvalue equations. The more complete problem, including $\mathbf{J}_0$ (which gives more terms on the above right hand sides), has also been implemented, but is not fully tested for yet, so results will primarily be presented for solutions to Eqs. (6) – (8). Calculations are first performed for a large aspect ratio circular plasma, to see that simple eigenmodes with a dominant poloidal mode number could be found, see Fig. 1. In this and the following simulations a parabolic density profile with $n = 3.5 \cdot 10^{19} \text{m}^{-3}$ on the magnetic axis was assumed.
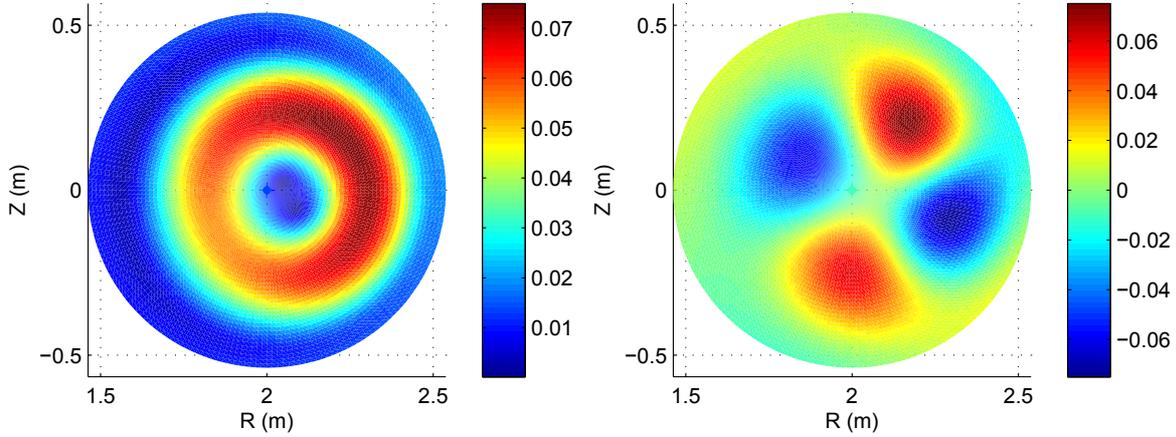
Figure 1: The numerical scheme is tested for a large aspect ratio ($R/a \simeq 3.6$) circular plasma ($n = 1$ is assumed). The magnitude $|b_\parallel|$ of the parallel magnetic field component (left) and a snapshot of $\mathrm{Re}\{b_\parallel e^{i(n\varphi - \omega t)}\}$ at a certain time t (right).

Next, tight aspect ratio tokamaks are studied. The flux coordinates, the metric tensor, the magnetic field and the current density of the underlying equilibrium are taken either from calculations with the CHEASE code [9] or from an analytical Solovev equilibrium. Figure 2 shows results obtained using a CHEASE equilibrium and Fig. 3 gives results from a simple Solovev equilibrium where the toroidal magnetic field is $\propto 1/R$. In the latter case, the $\mathbf{J}_0$ terms in Eq. (3) are included.
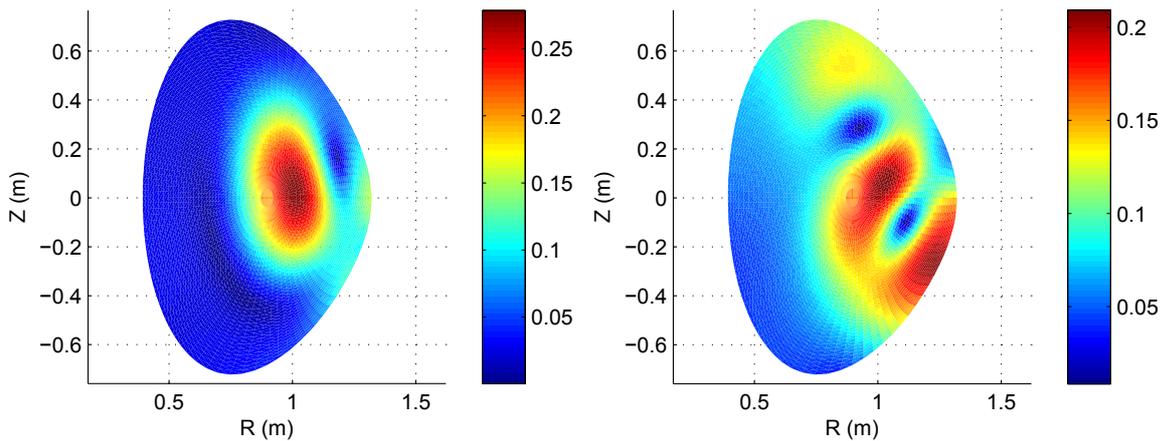

Figure 2: The magnitude of the parallel magnetic field component $|b_\parallel|$ for $n = 3$ solutions with $f = 1.71\,\mathrm{MHz}$ (left) and $f = 1.86\,\mathrm{MHz}$ (right). A MAST equilibrium from the CHEASE code [9] was used, where the ion cyclotron frequency ranges from 2.4 MHz on the outboard side to 7.4 MHz on the inboard side.
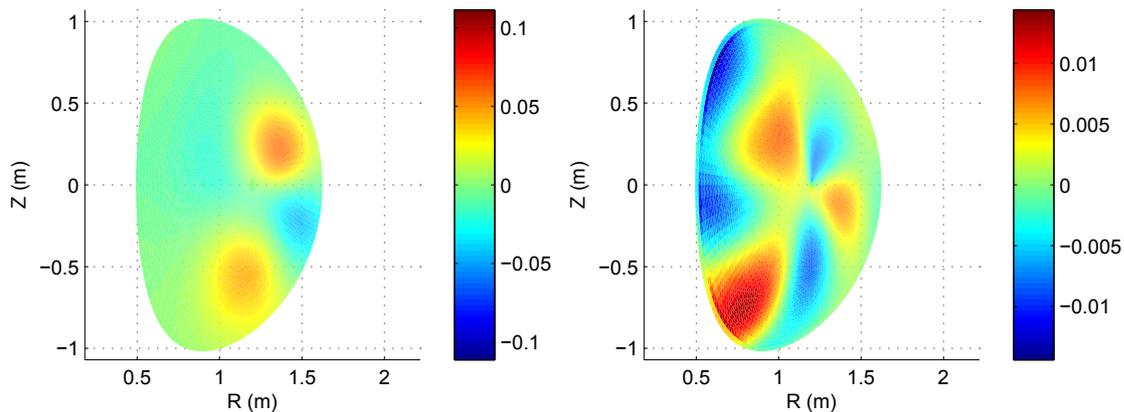
Figure 3: A solution for $n = 1$ with $f = 1.76$ MHz for an analytical Solovev equilibrium with ion cyclotron frequency in the range $3.0 - 8.5$ MHz. The parallel component $|b_{\parallel}|$ (left) and snapshots of $\mathrm{Re}\{b_{\parallel}e^{-i\omega t}\}$ (middle) and the much smaller $\mathrm{Re}\{B_r e^{-i\omega t}\}$ component (right) are shown. The full equations including $\mathbf{J}_0$ were solved.

It is generally found that the parallel component of the perturbed magnetic field is largest on the outboard side. Solutions for the high $|n|$ numbers (5–11) observed in experiments are not easily found. In the future, the numerical scheme will be improved to include the $\mathbf{J}_0$ terms, and a vacuum region will be added between the plasma and the conducting wall, to obtain more realistic boundary conditions. Finally, the code also needs to be compared with experimental and theoretical results.

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