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Tokamaks with Reversed Current Density: Stability of Equilibria with Axisymmetric Islands

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Tokamak regimes with nearly zero or negative toroidal current in the central region (current hole) feature good plasma confinement properties and may be naturally attained in the advanced tokamak (AT) regime. On the other hand, quasi-steady-state AC plasma current operation in tokamaks relies upon the existence of current reversal equilibria with zero net toroidal current. In both cases a system of axisymmetric magnetic islands exists in the equilibrium configurations. Recent papers on the reconstruction of current hole equilibria consistent with experimental data [1] and AC tokamak operation experiments [2] provide examples of experimentally relevant equilibria without nested flux surfaces.

The question of the stability of such configurations has received little attention so far. The great variety of topologies of these islands poses non trivial challenges for its numerical resolution. In particular, the common use of structured, magnetic surface-aligned grids is practically prevented. This paper deals with the axisymmetric stability using a newly developed code [3, 4] based on finite elements on an unstructured triangular grid. We show its ability to treat arbitrary island topologies.

1. Ideal MHD stability: the problem formulation and approximation

For the stability analysis the potential and kinetic energy functionals can be expressed in terms of the electric field perturbation $\vec{E} = i\omega\vec{e}, \vec{e} = -\vec{\xi} \times \vec{B}$:

$$W_p = \frac{1}{2}\int\left\{|\nabla\times\vec{e}|^2 - \frac{\vec{j}\cdot\vec{B}}{B^2}\vec{e}\cdot\nabla\times\vec{e} + \frac{\vec{j}\cdot\vec{e}}{B^2}\left[2\vec{B}\cdot\nabla\times\vec{e} - \vec{t}\cdot\vec{e}\right]\right\}d^3r, \quad (1)$$

$$K_p = \frac{1}{2}\int\rho|\vec{e}|^2/B^2 d^3r, \quad \vec{t} = \vec{j} + B^2\nabla(\frac{1}{B^2})\times\vec{B}, \quad (2)$$

combined with the requirement $(\vec{e}\cdot\vec{B}) = 0$. The surrounding vacuum region (free boundary) can also be taken into account (see [4]). For tokamak modelling we use the standard equilibrium magnetic field representation, $\vec{B} = \nabla\psi\times\nabla\phi + f\nabla\phi$, with the poloidal field function ψ solution of the Grad-Shafranov equation:

$$-R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = R^2 p' + f f' \quad (3)$$

with zero (force-free) or finite pressure gradient. For the force-free case the last term in the functional (1) vanishes.

The approach to approximate and solve the stability problem on triangular grids includes:

- different finite elements for the longitudinal e_ϕ and poloidal $\vec{e}_{pol}$ projections of the unknown vector $\vec{e} = e_\phi\nabla\phi + \vec{e}_{pol}$: standard node-based "hat"-functions W_i for e_ϕ, and edge-based Whitney elements W_{mn}, $W_{mn} = W_m\nabla W_n - W_n\nabla W_m$ for $\vec{e}_{pol}$;
- Lagrange multipliers introduced to approximate the constraint $(\vec{e}\cdot\vec{B}) = 0$ at each grid node in the plasma region;
- solution of the saddle point matrix eigenvalue problem.

A crucial point is the approximation of the constraint $(\vec{e}\cdot\vec{B}) = 0$. In particular, introducing a small regularizing term into the Lagrange multiplier formulation helps to get robust LU decomposition of the saddle point matrices with standard sparse matrix reordering.

The equilibrium and stability calculations use the same unstructured grids optionally adapted to the solution features (e.g. jump in the current density) [4]. For convergence studies triangulated structured grids (polar or rectangular) have been used with up to 160 nodes in each direction.

2. Free boundary tilt instability: infinite aspect ratio case

Let us consider a plasma equilibrium consisting of two oppositely directed current channels (dipole) imbedded into an external magnetic field [5–7]. The corresponding magnetic flux function ψ is analytically described inside and outside the unit circle (Fig.1a):

$$\psi = 2/(kJ_0(k))J_1(kr)sin\theta, \quad p' + ff' = k^2\psi, \quad r \leq 1, \tag{4}$$

$$\psi = (r - 1/r)sin\theta, \quad p' = ff' = 0, \quad r > 1, \tag{5}$$

is an exact solution of equation (3) with $R = const = 1$ (no toroidicity). Here k is the first root of the Bessel function J_1. The force balance in (4) can be provided either by a pressure gradient or a magnetic field component in the z direction, $B_z = f$. The value of B_z is set to 1.0 at $r = 1$, and a constant plasma density $\rho = 1$ is assumed. For stability calculations the standard boundary conditions $\vec{e}_\tau = 0$ are set at some chosen wall radius r_w. For the axisymmetric stability problem the currentless region, $r > 1$, is assumed to have zero plasma density and is therefore equivalent to vacuum. Our numerical studies show that the wall at infinity limit (growth rate saturation) is nearly reached for $r_w \geq 2$. Triangulated structured grids in polar coordinates are used. Large enough rectangular domains and uniform grids in Cartesian coordinates (x, y) give the same results but the convergence is slower.

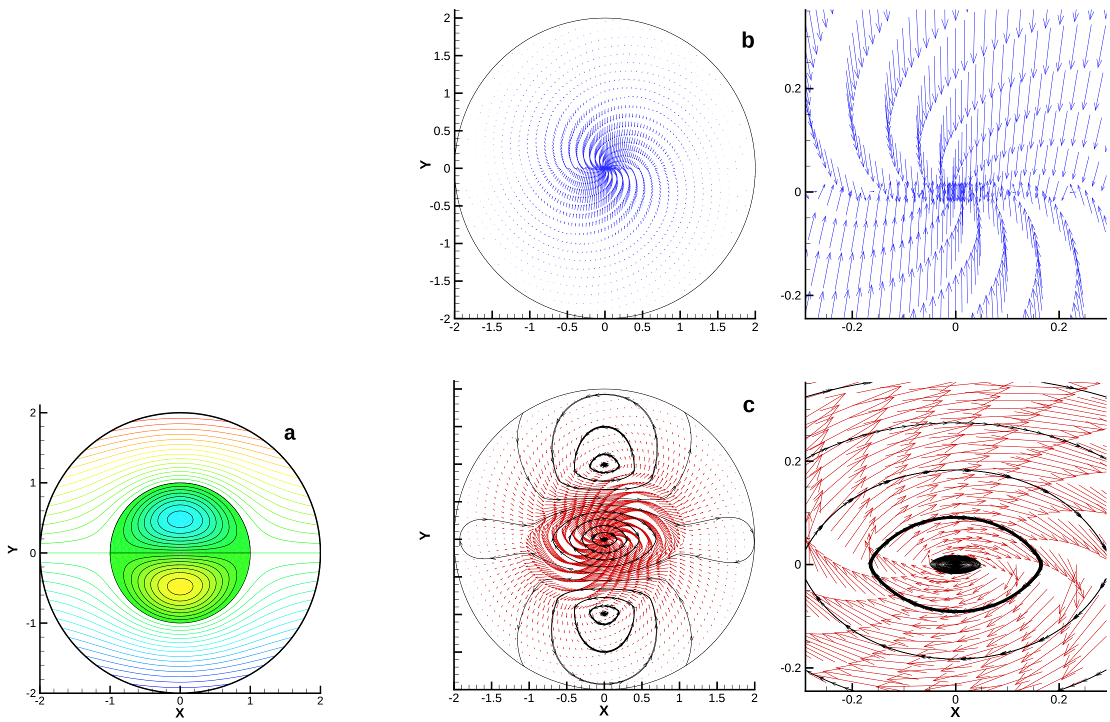

Fig.1. Tilt instability test: a) equilibrium flux function contours; b) eigenfunction $\vec{e}_{pol}$; c) the reconstructed displacement vector $\vec{\xi}_{pol}$ with streamlines.

In agreement with the results of [5] this dipole configuration exhibits a tilt instability: the current channels tend to rotate towards the equatorial plane, see Fig.1. The corresponding velocity field features a main vortex around the separatrix and two satellite vortices with smaller velocities. The growth rate of the instability somewhat increases with an increasing pressure gradient by choosing different p' and ff' for the same equilibrium from $\omega^2 = -2.8$ (force-free) up to $\omega^2 = -3.2$ for the highest pressure case with $ff' = 0$ in the entire domain. Some discrepancies with the results presented in [5–7] can be explained by different MHD models used. Anyway, the problem considered provides a good benchmark case for the MHD codes not assuming nested magnetic surfaces.

3. Axisymmetric stability of toroidal analytical reversed current density configurations

Stability calculations show that many of the reversed current density configurations are unstable against the $n = 0$ mode even with a conducting wall at the plasma boundary. Here the computational results are presented for analytically described toroidal axisymmetric configurations with rectangular cross-sections [8, 9], assuming the following profiles in the Grad-Shafranov equation (3): $p' = -a_1$, $ff' = a_2 + \alpha^2\psi$, and a given aspect ratio A and elongation E. Then the solution ψ can be represented as a series of the known eigenfunctions of the Grad-Shafranov operator: Bessel and cosine functions in R and Z directions respectively. About 60 expansion terms in each direction are retained to obtain an accurate solution on the uniform triangular grid. All the published examples with current reversal in square domains have been reproduced. Several configurations of dipole type with varied elongation were also considered to assess its role on their stability. It is found that the dipole type equilibria (Fig.2, left) featuring a magnetic island separatrix aligned with the short semi-axis of the plasma poloidal cross-section (total elongation less than unity) are stable, at least in the fixed boundary case.

Fig.2 (right) shows another example of the current density reversal configurations from [9] which are supposed to model tokamak AC operation when the total plasma current changes sign and passes zero. This equilibrium is also stable with a fixed boundary.

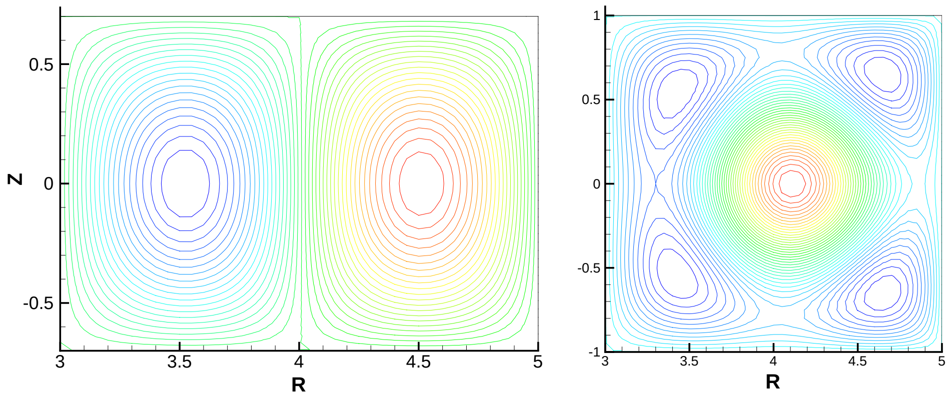

Fig.2. Two analytic solutions of the Grad-Shafranov equation (equilibrium flux function contours). Left: dipole type solution in an oblate cross-section, $E = 0.7$, $A = 4, a_1 = 0.00395, a_2 = 0.0685, \alpha = 3.9432$. *Right: central current density reversal with zero total current,* $A = 4, a_1 = 0.00259, a_2 = 0.0261, \alpha = 4.7083$. *Both configurations are stable against internal* $n = 0$ *modes.*

Unstable configurations are presented in Fig.3. The dipole equilibria are weakly unstable and the growth rate increases with elongation, $E > 1$. A similar behaviour was observed with smooth boundary shapes and other plasma profiles [3]. Roughly the same eigenfunction structure (main vertex with two satellites) as for the planar test (Fig.1) is observed in this fixed boundary case.

More complicated topology of the magnetic islands, Fig.3(lower), leads to rather patterned eigenfunctions. Finer grids are probably needed to resolve the details, but typically the maximal velocity corresponds to regions of small poloidal field in the outer island.

To conclude, the use of unstructured adaptive grids and a rather general approach to the approximation of the linear MHD equations open new capabilities for the stability analysis of plasma equilibria with islands. Further applications and developments of the method will include:

- pressure and current profile influence on n=0 stability;
- an extension of the approach to kink mode stability analysis.

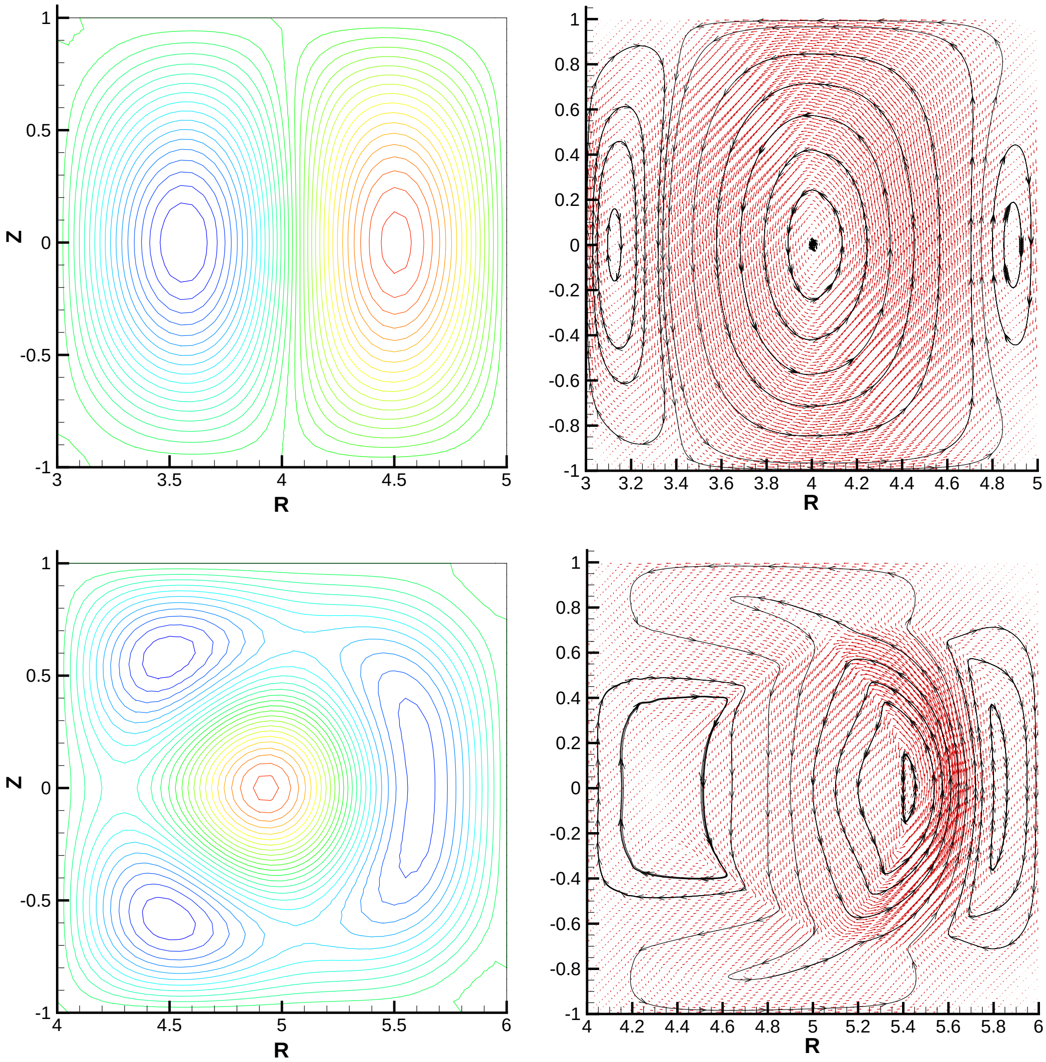

Fig.3. Two examples of unstable equilibria (left: contours of ψ) and eigenmode displacement vector $\vec{\xi}_{pol}$ (right: arrows and streamlines).
Upper row: $A = 4, a_1 = 0.00395, a_2 = 0.0685, \alpha = 3.9432,$ *eigenvalue* $\omega^2 = -1.3e-05.$
Lower row: $A = 5, a_1 = -0.06; a_2 = -0.0006; \alpha = 5.50,$ *eigenvalue* $\omega^2 = -9.4e-03.$

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