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JEANS THEOREM AND THE NUMBER OF INDEPENDENT CONSTANTS OF MOTION

O.Ågren[1], V.E.Moiseenko[1,2], A, Hagnestål[1]

[1] *Uppsala University, SE-751 21Uppsala, Sweden*

[2] *National Science Center "Kharkiv Institute of Physics and Technology"*

61108 Kharkiv, Ukraine

The three established adiabatic invariants are separating invariants in the sense of Liouville. Two additional independent (not separating) invariants can exist. We obtain a strengthened version of Jean's theorem, where all independent constants of motion are required to obtain the general solution of the stationary Vlasov equation [1]. The five time independent characteristic equations for the single particle orbit determine five independent constants of motion. In a constrained phase space with a global symmetry, the number of independent constants of motion is reduced. Four independent constants of motion exist in the constrained phase space of axisymmetric tori [1], while only three independent constants of motion can be used for to derive a screw pinch equilibrium. A radial drift invariant, corresponding to the average of the radial coordinate of the gyrating particle, is a constant of motion for a confined particle, and its relation to other more well known constants of motion is investigated [1,2].

The results above are derived from the stationary Vlasov equation,

$$\sum_{i=1}^{3}(\mathrm{v}_i \frac{\partial f}{\partial x_i} + \frac{F_i}{M}\frac{\partial f}{\partial \mathrm{v}_i}) = 0 \quad , \quad \mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{1a}$$

This is a set of first order differential equations, and its solutions are fully determined by the characteristics which could be derived from the set of five ordinary differential equations

$$\frac{dx}{\mathrm{v}_x} = \frac{dy}{\mathrm{v}_y} = \frac{dz}{\mathrm{v}_z} = \frac{Md\mathrm{v}_x}{F_x} = \frac{Md\mathrm{v}_y}{F_y} = \frac{Md\mathrm{v}_z}{F_z} \tag{1b}$$

This system can be solved directly, or alternatively by introducing a curve parameter τ, by which the set of equations can be written in the familiar form $d\mathbf{x}(\tau)/d\tau = \mathbf{v}(\tau)$ and $Md\mathbf{v}(\tau)/d\tau = \mathbf{F}[\mathbf{x}(\tau),\mathbf{v}(\tau)]$, which shows that the equations of the characteristics of the collisionless Boltzmann equation are determined by the particle orbits. The integrals of the five equations in (1b) determine *five* independent constants of motion $I_k(\mathbf{x},\mathbf{v})$ (and also one time dependent invariant).

For Eq. (1a), we restrict the analysis to a finite domain with a finite variation of the chosen phase space variables. By restricting the analysis to that kind of finite domain, certain intricate mathematical questions such as multi-valued functions can be avoided. If the derivatives $\partial F_i / \partial x_j$ are continuous, five independent time independent solutions

$$I_k(\mathbf{x},\mathbf{v}) = const \ , k = 1,..,5 \tag{2a}$$

exist for the system (1b). The functional independence of these constants of motion is proved by the fact that the functional matrix $\partial(I_1,...I_5)/\partial(\mathbf{x},\mathbf{v})$ has a rank 5. From this follows that any stationary solution of the kinetic equation for a point mass is a function of the five independent constants of motion, i.e.

$$f = f(I_1, I_2, I_3, I_4, I_5) \ . \tag{2b}$$

The original version of Jeans theorem is that the distribution function is a function of an unspecified number of invariants, but here the theorem is strengthened to the more specific result that the general solution of the stationary Vlasov equation is a function of five independent constants of motion.

Systems with global symmetry

A special treatment of the characteristic equations is required if a global symmetry is imposed. For a screw pinch satisfying the global symmetries $\partial f / \partial\theta = \partial f / \partial z = 0$ in the cylindrical coordinates (r,θ,z), there are *three* time independent characteristic equations,

$$\frac{dr}{\mathrm{v}_r} = \frac{M d\mathrm{v}_r}{q(E_r + \mathrm{v}_\theta B_z - \mathrm{v}_z B_\theta) + \dfrac{M\mathrm{v}_\theta^2}{r}} = \frac{-M d\mathrm{v}_\theta}{q\mathrm{v}_r B_z + \dfrac{M\mathrm{v}_r\mathrm{v}_\theta}{r}} = \frac{M d\mathrm{v}_z}{q\mathrm{v}_r B_\theta} \tag{3a}$$

This system has the three integrals $(\varepsilon, p_\theta, p_z)$, i.e. the energy and two canonical momenta. Since the system of equations (3a) has only three equations, the corresponding phase space is reduced by the imposed symmetries. Mathematical theory shows that these three integrals are the complete system of constants of motion for Eq. (3a). An analogous conclusion holds for slab geometry with only one non-ignorable coordinate.

For axisymmetric tori with the toroidal symmetry $\partial f / \partial\varphi = 0$, the kinetic equation reads

$$\frac{\partial f}{\partial r_0}(\mathbf{v}\cdot\nabla r_0) + \frac{\partial f}{\partial\zeta}(\mathbf{v}\cdot\nabla\zeta) + \breve{F}_{r_0}\frac{\partial f}{\partial\dot{r}_0} + \breve{F}_\zeta\frac{\partial f}{\partial\dot{\zeta}} + (\frac{F_\varphi}{M} - \frac{\mathrm{v}_R\mathrm{v}_\varphi}{R})\frac{\partial f}{\partial\mathrm{v}_\varphi} = 0 \tag{3b}$$

where $\dot{r}_0 = dr_0/dt$, $\dot{\zeta} = d\zeta/dt$, $r_0(R,Z)$ is the radial flux coordinate and $\zeta(R,Z)$ is the poloidal angle. The four independent characteristic equations determine *four* independent constants of motion [1]. The limit of a screw pinch, where the poloidal angle becomes a cyclic coordinate and $\partial f / \partial\zeta \to 0$, is a degenerate case where it is possible to define a reduced phase space with this global symmetry, and only three independent constants of motion are required to span up that constrained phase space.

The results above suggests that the fourth independent invariant for axisymmetric tori, which will be identified as a radial drift invariant, degenerates to a dependent invariant in the limit of a screw pinch [1]. In Ref. [2], the radial drift invariant is calculated from the projected gyro center orbit equations

$$\frac{d\bar{r}_0}{dt} = -\frac{\tilde{B}_t}{qB\tilde{B}}\frac{1}{\bar{r}_0}\Big[\ \frac{\partial\hat{U}}{\partial\bar{\zeta}} + 2(\varepsilon-\hat{U})\frac{1}{B}\frac{\partial B}{\partial\bar{\zeta}}\ \Big] \tag{4a}$$

$$\frac{d\bar{\zeta}}{dt} = \frac{\tilde{B}_p}{\tilde{B}}\frac{\mathrm{v}_{\parallel}}{\bar{r}_0} + \frac{\tilde{B}_t}{qB\tilde{B}}\frac{1}{\bar{r}_0}\Big[\ \frac{\partial\hat{U}}{\partial\bar{r}_0} - 2(\varepsilon-\hat{U})\frac{c_0(r_0,\zeta)}{\bar{r}_0}\ \Big] \tag{4b}$$

where $\hat{U}(\bar{r}_0,\bar{\zeta}) = q\bar{\phi} + \mu\bar{B}$ is the guiding center potential, the energy $\varepsilon = \hat{U} + m\mathrm{v}_{\parallel}^2/2$ is constant, $\sigma = \pm 1$ determines the direction of the parallel velocity, and the expression for $c_0(r_0,\zeta)$ is given in Ref. [2]. In the limit of a screw pinch, $c_0(r_0,\zeta) \to \tilde{B}_p^2/B^2$, and $\bar{r}_0$ and $d\bar{\zeta}/dt$ are constants. For trapped particles in axisymmetric tori fields, $d\bar{\zeta}/dt$ changes sign.

The Poincare-Bendixson theorem shows that the projected gyro center orbit is a closed curve. For trapped particles, two orbit portions $\bar{r}_0(\bar{\zeta}) \equiv \bar{r}_0(\sigma,\varepsilon,\mu,\bar{\zeta})$ with opposite signs of $\mathrm{v}_{\parallel}$ connect at the points where σ changes sign. For axisymmetric tori with up-down symmetry, the closed projected gyro center orbit gives a radial drift invariant I_r in the form [2],

$$I_r(\sigma,\varepsilon,\mu,\bar{r}_0,\bar{\zeta}) = \bar{r}_0 + I_r^{(1)} = \bar{r}_0 + \sum_{n=1}^{\infty} I_{r,n}^{(1)}(\sigma,\varepsilon,\mu,\bar{r}_0)\cos\frac{n\pi\bar{\zeta}}{\bar{\zeta}_b} \tag{4c}$$

where $\bar{\zeta}_b = \pi$ for passing particles and $\bar{\zeta}_b < \pi$ for trapped particles and $I_r^{(1)}$ determines the size of the guiding center displacements from the mean flux surface of the guiding center motion. This single valued constant of motion cannot be reduced to a dependent invariant (except in the screw pinch limit), since I_r has a different dependence on the poloidal angle than the constants of motion $(\varepsilon,\mu,I_\varphi)$, which is most clearly seen for trapped particles. Distribution functions of the four independent constants of motion, i.e.

$$F(\varepsilon,\mu,I_\varphi,I_r),$$

can therefore be used to describe axisymmetric tori equilibria. Of particular interest is a nearly local Maxwellian distribution function expressed in the radial drift invariant, i.e.

$$F(\varepsilon,\mu,I_\varphi,I_r) \;=\; n_0(I_r)\,[\frac{2/m}{\pi k_B T_0(I_r)}]^{3/2}\,e^{-\varepsilon/k_B T_0(I_r)} + F^{(1)}$$

where $F^{(1)}$ is required to obtain a toroidal current and a rotational transform and $n_0(r_0)$ and $T_0(r_0)$ corresponds to prescribed functions for the density and temperature. If the contribution from $F^{(1)}$ to the $\mathbf{j}\times\mathbf{B}$ force and the pressure is assumed small, this results in the force relation $\mathbf{j}\times\mathbf{B} = -\nabla P$ and thereby corresponds to the same equilibrium as in the simplified fluid description of ideal MHD. This form of the distribution function is also consistent with the starting point for neoclassical transport theory, where the guiding center part of the distribution function to leading order is assumed constant on a magnetic flux surface.

The result can be strengthened for a screw pinch [1], where radial profiles for the density, temperatures, parallel current, parallel momentum and scalar electric potential can be prescribed independently in a nearly Maxwellian distribution function [1].

$$F_\pm(\varepsilon,\mathrm{v}_\parallel,\bar{r}) = n_o(\bar{r})[\frac{M/2}{\pi k_B T_{0,\pm}(\bar{r})}]^{3/2}\exp[-\frac{\varepsilon'}{k_B T_{0,\pm}(\bar{r})}]$$

where a Galilei-transformed energy is introduced,

$$\varepsilon'(\varepsilon,\mathrm{v}_\parallel,\bar{r}) = q[\phi(r)-\phi_0(\bar{r})] + \frac{M}{2}\mathrm{v}_\perp^2 + \frac{M}{2}[\mathrm{v}_\parallel - \mathrm{v}_{0,\pm}(\bar{r})]^2$$

and the functions $n_0, \phi_0, T_{0,\pm}$ and $\mathrm{v}_{0,\pm}$ are prescribed radial profiles for the density, electric potential, ion and electron temperatures and the mean velocities along **B**. Use has been made of the facts that in a screw pinch $I_r = \bar{r}$ and the guiding center parallel velocity is constant. In a small gyro radius expansion, quasi neutrality is assured and the magnetic field components are determined by

$$B_z(r) \;=\; B_0 - \mu_0 \int_r^\infty dr\,(\frac{B_\theta}{B} j_\parallel \;+\; \frac{B_z}{B^2}\frac{dP}{dr})$$

$$B_\theta^2(r) \;=\; -\frac{1}{r^2}\int_0^r dr\, r^2 \frac{d}{dr}[\,B_z^2 + 2\mu_0 P(r)\,]$$

where $j_\parallel(r) = n_0(r)\sum_i q_i \mathrm{v}_{0,i}(r)$ and $P = n_0(r)\sum_i k_B T_{0,i}(r)$. The last equation is identical to the radial force balance of ideal MHD.

References

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