

Time dependent solutions of collisional electron kinetic equation

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Introduction. In many important cases (e.g. ELM burst propagation into SOL) the applicability of a fluid description may be questionable and one should treat plasma transport kinetically. Usually it can be done only numerically, which is very difficult and time consuming. In addition, complex nonlinear kinetic codes require benchmarking, which is not a trivial problem on its own. However, in some cases it is possible to find analytic solutions of nonlinear kinetic equation, which exhibit some important features of the problem of interest and help to benchmark kinetic codes. In particular in Ref. 1 it was shown that electron kinetic equation

$$\partial_t f = C(f, f) + H(f) \quad (1)$$

which can be used as a simplified model for ELM dynamic, has meaningful and useful analytic solutions. Here $f(\epsilon, t)$ is the symmetric electron distribution function, ϵ is the variable „energy,, $C(f, f)$ is the Coulomb collision term, $H(f)$ is the heating term

$$H(f) = \epsilon^{-1/2} \partial_\epsilon (D(\epsilon, t) \partial_\epsilon f), \quad (2)$$

and $D(\epsilon, t)$ is the effective diffusion coefficient in the energy space. The solutions found in [1] have self-similar form

$$f(\epsilon, t) = n(t) (T(t))^{-\alpha} F(\epsilon/T(t)), \quad (3)$$

where $n(t)$ is the electron density and $T(t)$ is an effective electron temperature defined as

$$T(t) = (2/3) \int f(\epsilon, t) \epsilon^{3/2} d\epsilon / \int f(\epsilon, t) \epsilon^{1/2} d\epsilon, \quad (4)$$

and α is an adjustable parameter. The ansatz (3) conserves the electron density $n = \int f(\epsilon, t) \epsilon^{1/2} d\epsilon$ in time for $\alpha = 3/2$. Therefore in what follows we will assume that $\alpha = 3/2$. Self-similar solutions are possible for $D(\epsilon, t) = D_\xi(\xi) T(t)$, where $\xi = \epsilon/T(t)$ and $D_\xi(\xi)$ is an arbitrary function. However, $D_\xi(\xi)$ considered in [1] were localized at small ξ . As a result, the solutions of the kinetic equation (1) have strongly depleted tails in comparison to Maxwellian one. Such feature seems to be inconsistent with ELM dynamics in SOL, where one expects fast particles to arrive first and to elevate the tail. Therefore in this paper we extend the results of [1] for a broader class of $D_\xi(\xi)$ resulting in enhancement of the tail. We find asymptotic solutions of (1) analytically and verify them numerically.

Analytic solutions of self-similar kinetic equations. Following Ref. 1 we consider the solutions of Eq.(1) in the self-similar form. Substituting (3) in (1) and using self-similar variable $\xi = \epsilon/T(t)$ we find

$$-(T^{1/2} / \hat{v}_c) (dT/dt) \xi^{-1/2} d_\xi (\xi^{3/2} F) = C_\xi(F, F) + \xi^{-1/2} d_\xi \{ \{D(\epsilon, t) / \hat{v}_c T(t)\} d_\xi F \}, \quad (5)$$

where

$$C_\xi(F, F) = \xi^{-1/2} d_\xi (A_F F + B_F d_\xi F), \quad A_F = \int_0^\xi d\xi \xi^{1/2} F(\xi) / \int_0^\infty d\xi \xi^{1/2} F(\xi),$$

$$B_F = 2 \left\{ \int_0^\xi d\xi \xi^{3/2} F(\xi) + \int_\xi^\infty d\xi \xi^{3/2} F(\xi) \right\} / 3 \int_0^\infty d\xi \xi^{1/2} F(\xi), \quad (6)$$

and $\hat{v}_c = 2^{3/2} \pi e^4 \ln \Lambda / m^{1/2}$. We notice that both A_F and B_F are approaching unity for $\xi \rightarrow \infty$. One finds it by recalling anzatz (4), which gives

$$2 \int_0^\infty d\xi \xi^{3/2} F(\xi) / 3 \int_0^\infty d\xi \xi^{1/2} F(\xi) = 1. \quad (7)$$

Since the self-similar Coulomb collision term $C_\xi(F,F)$ does not depend on time explicitly, assuming

$$D(\varepsilon, t) = \hat{D}_\xi(\xi) \hat{D}_T(T(t)), \quad (8)$$

from (5) we find that in accordance with [1] for arbitrary function $\hat{D}_\xi(\xi)$ the self-similar solution is only possible for $\hat{D}_T(T(t)) = T(t)$ and

$$\gamma \equiv (T^{1/2} / \hat{v}_c) (dT/dt) = \text{const}. \quad (9)$$

In this case Eq. (5) can be reduced to

$$d_\xi \left\{ \left(A_F + \gamma \xi^{3/2} \right) F + \left(B_F + \hat{D}_\xi(\xi) / \hat{v}_c \right) d_\xi F \right\} = 0. \quad (10)$$

Since the operator $C_\xi(F,F)$ conserves self-similar energy ($\int_0^\xi C_\xi(F,F) \xi^{3/2} d\xi = 0$), from Eq. (5) we find the following relation between γ and $\hat{D}_\xi(\xi)$:

$$\gamma = - \frac{\int_0^\infty d\xi \frac{\hat{D}_\xi(\xi)}{\hat{v}_c} \frac{dF}{d\xi}}{\int_0^\infty d\xi \xi^{3/2} F} \equiv \frac{\int_0^\infty d\xi \frac{d\hat{D}_\xi(\xi)}{\hat{v}_c d\xi} F}{\int_0^\infty d\xi \xi^{3/2} F}. \quad (11)$$

Taking $F(\xi \rightarrow \infty) \rightarrow 0$ from (12) we derive a formal solution (see Ref. 1)

$$F(\xi) = F(0) \exp \left\{ - \int_0^\xi d\xi \left(\frac{A_F + \gamma \xi^{3/2}}{B_F + \hat{D}_\xi(\xi) / \hat{v}_c} \right) \right\}, \quad (12)$$

(we recall that both A_F and B_F are the functionals of F). In Ref. 1 both analytic and numerical analysis of kinetic equations (1) and (5) were performed under assumption that the function $\hat{D}_\xi(\xi)$ quickly goes to zero for $\xi \geq 1$. In this case asymptotic expression $F(\xi \rightarrow \infty)$ is

$$F(\xi \rightarrow \infty) \propto \exp \left\{ -\xi - (2/5) \gamma \xi^{5/2} \right\}. \quad (13)$$

As we see from the expression (13), the distribution function $F(\xi)$ has strongly depleted tail (in comparison with Maxwellian distribution function). For a small departure of the bulk of $F(\xi)$ from Maxwellian one, the magnitude of γ can be expressed in terms of $\hat{D}_\xi(\xi)$ by using integral expression (11):

$$\gamma = (4/3\sqrt{\pi}) \int_0^\infty d\xi \hat{D}_\xi(\xi) e^{-\xi} / \hat{v}_c. \quad (14)$$

However, the situation changes drastically when $\hat{D}_\xi(\xi)$ increases with increasing ξ . For example, taking $\hat{D}_\xi(\xi) = D_0 \xi^p$, where D_0 is the normalization constant and p is an adjustable parameter from Eq. (12) we find that for $5/2 > p > 3/2$, the distribution function $F(\xi)$ has strongly enhanced tail (in comparison with Maxwellian distribution function):

$$F(\xi \rightarrow \infty) \propto \exp \left\{ - \frac{\gamma \hat{v}_c \xi^{5/2-p}}{D_0 (5/2-p)} \right\} \quad (15)$$

Similar to (14), for a small departure of the bulk of $F(\xi)$ from Maxwellian one, the magnitude of γ can be expressed in terms of D_0 and p by using integral expression (11). As a result we find:

$$\frac{\gamma \hat{v}_c}{D_0} = \frac{\Gamma(p+1)}{\Gamma(5/2)}, \text{ and } F(\xi \rightarrow \infty) \propto \exp\left\{-\frac{\Gamma(p+1)}{\Gamma(5/2)} \frac{\xi^{5/2-p}}{(5/2-p)}\right\}, \quad (16)$$

where $\Gamma(x)$ is the Gamma function.

For $p > 5/2$ the self-similar solution of Eq. (1) is not valid because Eq. (12) gives diverging function $F(\xi)$: $F(\xi \rightarrow \infty) \rightarrow \infty$. However for $p = 5/2$ the expression (11) gives exact relation between γ , D_0 , and p , $\gamma \hat{v}_c / D_0 = 5/2$, regardless the function $F(\xi)$ (which is obvious since $d\hat{D}_\xi(\xi)/d\xi \equiv dD_0\xi^{5/2}/d\xi = (5/2)D_0\xi^{3/2}$ and we have the same integral $\int_0^\infty d\xi \xi^{3/2} F$ in both nominator and dominator of Eq. (11)), while Eq. (12) gives a power-law tail of $F(\xi)$:

$$F(\xi \rightarrow \infty) \propto \xi^{-\gamma \hat{v}_c / D_0} \propto \xi^{-5/2}. \quad (17)$$

The validity of such solution will depend on the convergence of the integral $\int_0^\infty d\xi \xi^{3/2} F$, which enters in the expressions of Eq. (5). As one sees from Eq. (17), the integral $\int_0^\infty d\xi \xi^{3/2} F$ diverges logarithmically at $\xi \rightarrow \infty$. Therefore the self-similar solution is not possible for $\hat{D}_\xi(\xi) = D_0 \xi^{5/2}$ for all values of ξ . However, we can have $\hat{D}_\xi(\xi \rightarrow \infty) \rightarrow D_0 \xi^{5/2}$, which would give $F(\xi \rightarrow \infty) \propto \xi^{-\gamma \hat{v}_c / D_0}$, but try to adjust $\hat{D}_\xi(\xi)$ at „small,, ξ , departing from $\hat{D}_\xi(\xi) \propto \xi^{5/2}$, in order to have $\gamma \hat{v}_c / D_0 > 5/2$, which is needed for convergence of $\int_0^\infty d\xi \xi^{3/2} F$. As one can see from Eq. (11) the inequality $\gamma \hat{v}_c / D_0 > 5/2$ and asymptotic $\hat{D}_\xi(\xi \rightarrow \infty) \rightarrow D_0 \xi^{5/2}$ will be satisfied if

$$D_0^{-1} d\hat{D}_\xi(\xi)/d\xi^{5/2} = g(\xi) \geq 1, \quad (18)$$

where $g(\xi \rightarrow \infty) \rightarrow 1$. As an example, we can take $2 \geq g(\xi) \equiv (2 + \xi^{5/2})/(1 + \xi^{5/2}) \geq 1$, which results in $\hat{D}_\xi(\xi) = D_0 \{\xi^{5/2} + \ln(1 + \xi^{5/2})\}$ and gives $\gamma \hat{v}_c / D_0 > 5/2$, insuring convergence of $\int_0^\infty d\xi \xi^{3/2} F$ and applicability of self-similar solution. However, in order to find exact value of the ratio $\gamma \hat{v}_c / D_0$ and asymptotic $F(\xi \rightarrow \infty)$, one needs to find solution of the kinetic equation for $0 < \xi < \infty$, which can be done only numerically even for a small departure of the bulk distribution from Maxwellian one.

Next we consider the case $\hat{D}_\xi(\xi) = D_0 \xi^p$ and $p = 3/2$. In this case, as one finds from Eqs. (7, 11), $\gamma \hat{v}_c / D_0 = 1$ and

$$F(\xi \rightarrow \infty) \propto \exp(-\xi). \quad (19)$$

It suggests that for the case $\hat{D}_\xi(\xi) = D_0 \xi^{3/2}$, $F(\xi)$ is Maxwellian for all ξ . To verify it let us take $D(\varepsilon, t) = D_0 \xi^{3/2} \hat{D}_T(T(t))$ and assume that $F(\xi)$ is Maxwellian. Then in Eq. (5) $C_\xi(F, F) \equiv 0$ and that Eq. (5) can be reduced to the equation for $T(t)$:

$$d_t T = D_0 \hat{D}_T(T(t)) T^{-3/2}. \quad (20)$$

Thus, for the case $\hat{D}_\xi(\xi) = D_0 \xi^{3/2}$ the distribution function stays Maxwellian even though the temperature changes in time in accordance with Eq. (22) and the applicability of self-similar solution is not restricted anymore by the assumption $\hat{D}_T(T(t)) = T(t)$.

However, it is not clear that the solution $f(\varepsilon, t) \propto \exp(-\varepsilon/T(t))$ and $T(t)$ from Eq. (20) is stable against small perturbations which will turn on Coulomb collision term, which due to synergetic effects of the heating may cause further departure from a local Maxwellian distribution function. Further studies, including numerical one, are needed to answer stability question.

Numerical solution of time-dependent kinetic equation. In order to verify our analytic results we solve Eq. (1) numerically. In numerical modeling we utilize finite difference schemes developed in [2], which conserve density and energy of the system and allow large time steps and velocity intervals without error accumulation. The results of the solutions are shown in Fig. 1 for the diffusion coefficient corresponding $p=3/2$, $7/4$, 2 , and $5/2$ (here $e(t) = 3T(t)/m$). As we see, the numerical solutions confirm the analytic results predicting enhanced asymptotic tail of the distribution functions. They also demonstrate that the Maxwellian solution $f(\varepsilon, t) \propto \exp(-\varepsilon/T(t))$ with $T(t)$ from Eq. (20) corresponding for $p=3/2$ is stable. Comparison of the tails of the distribution functions for the cases $p=3/2$, $7/4$, and 2 with analytic expression (16) shows good quantitative agreement. However, for the case $p=5/2$, the far tail of the distribution function does not obey the asymptotic (17), although still has the power law character. The reason for this is a „wrong,, form of the diffusion coefficient used in the modeling, which cannot satisfy the conservation laws (see discussions above).

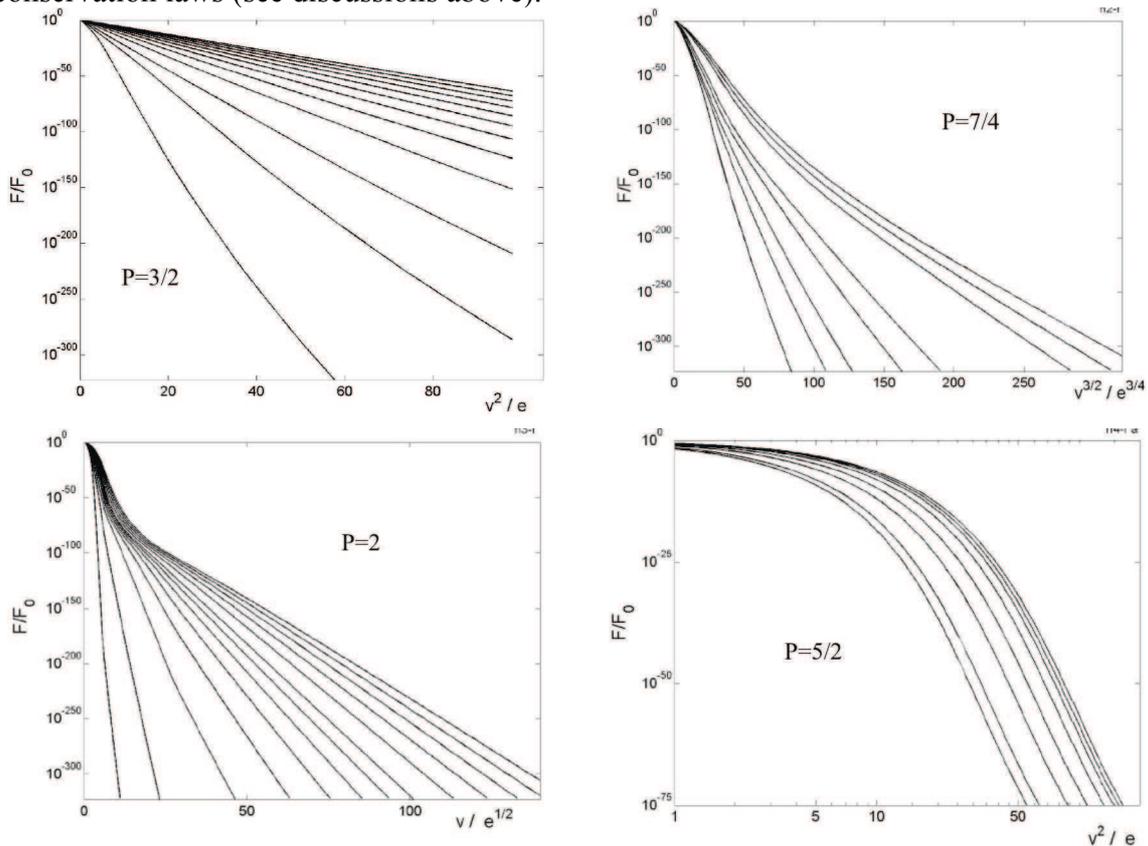


Fig. 1. The results of numerical solution of Eq. (1).

References

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