Alfvén eigenmodes in magnetic X-point configurations with strong longitudinal fields

N. Ben Ayed[1,2], K. G. McClements[2], A. Thyagaraja[2].

[1] *University of York, Department of Physics, Heslington, York, YO10 5DD, U.K.*
[2] *UKAEA Fusion Association, Culham Science Centre, Abingdon, Oxfordshire,U.K.*

1.   Introduction

The complex magnetohydrodynamic (MHD) properties of magnetic X-points have attracted much attention, due to the fact that configurations with this type of topology are common to many laboratory and natural plasma environments, and play a key role in processes such as magnetic reconnection and mode conversion. One such environment is the tokamak plasma in which divertor X-points are produced either by currents in the external coils, or in the plasma itself in the form of magnetic islands; in both cases, the effects of the X-point on nearby propagating Alfvén waves is still unknown. The fast magnetoacoustic wave is of particular importance in this context since, unlike the shear Alfvén wave, it can carry free magnetic energy into the X-point itself where it can be efficiently transformed into heat, mass flows or energetic particles [1,2]. The shear wave is also of interest, however, since, except for the special case in which there is no longitudinal field $B_\parallel$ and no variations in that direction, the presence of an X-point causes the shear and fast waves to be coupled [3]. This coupling was studied analytically in [4] for the case of an X-point with zero equilibrium current but finite $B_\parallel$, variations in the longitudinal direction being neglected. They showed that the coupling is associated with the formation of singular structures in the current density near the X-point separatrix. McClements and co-workers [5] solved the initial value problem of a fast wave being driven up by a shear wave for the case in which $B_\parallel$ is smaller than $B_\perp$, the field in the X-point plane, taking into account resistivity and electron inertial effects; again, variations in the longitudinal direction were neglected. In this contribution, a three dimensional perturbative analysis is presented of Alfvén waves in a magnetic X-point configuration with a strong longitudinal guide field.

2.   Equilibrium field

We consider a uniform equilibrium field, $\mathbf{B}_E = B_\parallel \hat{\mathbf{z}}$, where $B_\parallel$ is a constant and $\hat{\mathbf{z}}$ is the unit vector in the z direction. Transverse incompressible (shear Alfvén) waves propagating along $\hat{\mathbf{z}}$ are described by functions $f(x, y, z \pm c_A t)$ where $c_A^2 = B_\parallel^2/\mu_0 \rho_0$, μ_0 being the permeability of free space and ρ_0 the equilibrium plasma density, assumed to be uniform.

We now consider the effect of adding a current-free X-point field to this equilibrium:

$$\mathbf{B}_E = B_\parallel \hat{\mathbf{z}} + \frac{B_\perp}{r_0} \left(y\hat{\mathbf{x}} + x\hat{\mathbf{y}} \right), \tag{1}$$

where $B_\perp$, r_0 are constants and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ denote unit vectors in the x and y directions. The (x, y) components of the equilibrium field can be obtained from the flux function $\psi_E = \frac{B_\perp}{2r_0} \left(y^2 - x^2 \right)$. We take the limit $B_\perp/B_\parallel \equiv \lambda \ll 1$ and write the total equilibrium field in the non-dimensional form

$$\mathbf{b}_E = \mathbf{b}^{(0)} + \lambda \mathbf{b}^{(1)}, \tag{2}$$

where $\mathbf{b}_E = \mathbf{B}_E/B_\parallel$, $\mathbf{b}^{(0)} = \hat{\mathbf{z}}$ and $\mathbf{b}^{(1)} = (y\hat{\mathbf{x}} + x\hat{\mathbf{y}})/r_0$. Throughout this paper superscripts (0) and (1) denote respectively zeroth and first order quantities.

3.   Perturbative analysis

In the limit of zero plasma pressure and equilibrium flow the linearised ideal MHD momentum and induction equations are

$$\partial_t \tilde{\mathbf{V}} = (\rho_0 \mu_0)^{-1} \left(\nabla \times \tilde{\mathbf{B}} \right) \times \mathbf{B}_E, \qquad \partial_t \tilde{\mathbf{B}} = \nabla \times (\tilde{\mathbf{V}} \times \mathbf{B}_E),$$

where $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{V}}$ are the perturbations to the magnetic field and fluid velocity. We normalise the space coordinates x, y and z to r_0, $\tilde{\mathbf{B}}$ to $B_\parallel$, $\tilde{\mathbf{V}}$ to c_A, and t to r_0/c_A. We also express the velocity and field perturbations as power series in λ:

$$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}^{(0)} + \lambda \tilde{\mathbf{V}}^{(1)} + \lambda^2 \tilde{\mathbf{V}}^{(2)} + ... \qquad \tilde{\mathbf{B}} = \tilde{\mathbf{B}}^{(0)} + \lambda \tilde{\mathbf{B}}^{(1)} + \lambda^2 \tilde{\mathbf{B}}^{(2)} + ...$$

The mathematical steps are briefly summarised, and the reader is reffered to [6] for full details. The zeroth order equations are obtained by equating terms that are independent of λ:

$$\partial_t \tilde{\mathbf{V}}^{(0)} = (\nabla \times \tilde{\mathbf{B}}^{(0)}) \times \mathbf{b}^{(\mathbf{0})}, \qquad \partial_t \tilde{\mathbf{B}}^{(0)} = \nabla \times (\tilde{\mathbf{V}}^{(0)} \times \mathbf{b}^{(\mathbf{0})}).$$

The first order equations take the form:

$$\partial_t \tilde{\mathbf{V}}^{(1)} = (\nabla \times \tilde{\mathbf{B}}^{(0)}) \times \mathbf{b}^{(\mathbf{1})} + (\nabla \times \tilde{\mathbf{B}}^{(0)}) \times \mathbf{b}^{(\mathbf{0})}, \quad \partial_t \tilde{\mathbf{B}}^{(1)} = \nabla \times (\tilde{\mathbf{V}}^{(0)} \times \mathbf{b}^{(\mathbf{1})}) + \nabla \times (\tilde{\mathbf{V}}^{(\mathbf{1})} \times \mathbf{b}^{(\mathbf{0})}) \quad (3)$$

We introduce two new variables, $\chi^{(1)}$ and $\Lambda^{(1)}$, defined respectively as the divergence of $\tilde{\mathbf{V}}^{(1)}$ in the X-point plane ($\chi^{(1)} \equiv \partial_y \tilde{V}_y^{(1)} + \partial_x \tilde{V}_x^{(1)}$) and the longitudinal component of the first order vorticity, ($\Lambda^{(1)} \equiv \partial_y \tilde{V}_x^{(1)} - \partial_x \tilde{V}_y^{(1)}$). Furthermore, we introduce a stream function $A(x,y)$ for the leading order flow such that $\tilde{V}_x^{(0)} = \partial_y A \exp[i(k_z z - \omega t)]$, and $\tilde{V}_y^{(0)} = -\partial_x A \exp[i(k_z z - \omega t)]$. The component forms of leading and first order equations can be rearranged to give two equations for $\chi^{(1)}$ and $\Lambda^{(1)}$:

$$\left(\omega^2 - k_z^2 \right) \chi^{(1)} + \nabla_\perp^2 \chi^{(1)} = ik_z (y\partial_y - x\partial_x) \nabla_\perp^2 A.$$
$$- \left(\omega^2 - k_z^2 \right) \Lambda^{(1)} = 2ik_z \left[(x\partial_y + y\partial_x)\nabla_\perp^2 A + 2\partial_{xy}^2 A \right]$$

Imposing the shear Alfvén dispersion relation $w = k_z$ (recovered from leading order equations), we find that these equations reduce to:

$$(x\partial_y + y\partial_x) \nabla_\perp^2 A + 2\partial_{xy}^2 A = 0 \qquad \nabla_\perp^2 \chi^{(1)} = ik_z (y\partial_y - x\partial_x) \nabla_\perp^2 A.$$

Fourier transforming in x and y, putting $k_x = k_\perp \cos\theta$, $k_y = k_\perp \sin\theta$ and letting $u = \cos 2\theta$, we obtain:

$$(k_\perp \partial_{k_\perp} - 2u\partial_u) \hat{A}(k_\perp, u) = -\hat{A}(k_\perp, u) \qquad \hat{\chi}^{(1)}(k_\perp, u) = i\omega \left[uk_\perp^{-1}\partial_{k_\perp} \left(k_\perp^2 \hat{A} \right) + 2 \left(1 - u^2 \right) \partial_u \hat{A} \right]$$

where $\hat{A}(k_x, k_y) = 1/(4\pi) \int \int A(x,y) \exp\left[-i(k_x x + k_y y) \right] dxdy$ and $\hat{\chi}^{(1)}$ is determined in a similar way.

4.   Solutions

The general solution for $\hat{A}$ is readily determined using the method of characteristics and can be written in the form:

$$\hat{A}(k_\perp, u) = \frac{1}{k_\perp} H(k_\perp^2 u) = \frac{1}{k_\perp} H(k_\perp^2 \cos 2\theta),$$

where H is an arbitrary function. Taking the inverse Fourier transform and setting $x = r \cos\varphi$, $y = r \sin\varphi$ we deduce that the general solution for the zeroth order stream function is

$$A(r, \varphi) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} H(k_\perp^2 \cos 2\theta) \exp\left[ik_\perp r \cos(\theta - \varphi) \right] dk_\perp d\theta.$$

For the case $H = 1$, it is found that $A(r, \phi) = 1/r$. The zeroth order flows are $\tilde{V}_x^{(0)} = -(\sin\varphi/r^2)\exp[ik_z(z-t)]$, $\tilde{V}_y^{(0)} = (\cos\varphi/r^2)\exp[ik_z(z-t)]$. Zeroth order streamlines for flow and field ($\tilde{B}_x^{(0)} = -\tilde{V}_x^{(0)}$, $\tilde{B}_y^{(0)} = -\tilde{V}_y^{(0)}$) are circles centred around the X-line, but singular at $r = 0$.

It is also found that the first order compressibility can be expressed as $\hat{\chi}^{(1)} = -i\omega\cos 2\varphi/r$, and thus has the same angle dependence as the equilibrium flux function. The first z-components of the flow and field have the azimuthal structure, $\tilde{V}_z^{(1)} = \tilde{B}_z^{(1)} = -\cos 2\varphi/r$. The fact that these quantities are finite indicates that the wave is no longer polarised in the plane perpendicular to the propagation direction, [2,3].

The singularity at $r = 0$ in the previous solution can be avoided by choosing a different function H. For example, we can set:

$$\begin{aligned} H &= \exp\left[-k_\perp^2\cos 2\theta\right], \quad \cos 2\theta > 0, \\ &= 0, \qquad\qquad\qquad \cos 2\theta < 0. \end{aligned}$$

The double integral for $A(r, \phi)$ (eq. 4) can be reduced analytically. Putting $r = 0$ we find that this integral converges and has the value $A = \Gamma(5/4)/\Gamma(3/4) \simeq 0.74$.

Shown in figure 1 (a) is a plot of the contours of A in the (x, y) plane, representing the streamlines of the zeroth order flow. A striking contrast is immediately apparent between streamlines
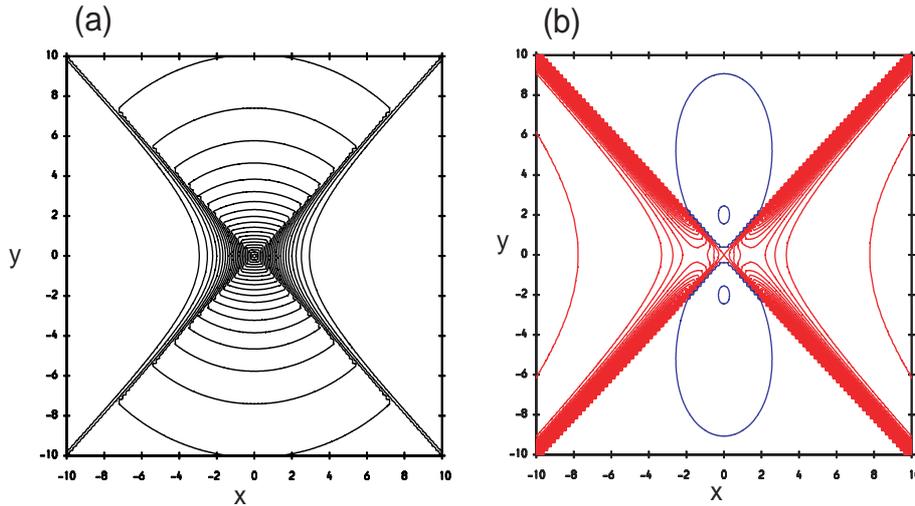

Figure 1: *Contour plots of (a) $A(x,y)$, i.e, streamlines of the zeroth order flow, when $H = \exp[-k_\perp^2\cos 2\theta]$ for $\cos 2\theta > 0$, $H = 0$ for $\cos 2\theta < 0$. (b) $\tilde{V}_z^{(1)}(x, y)$ when $H = k_\perp^2\cos 2\theta$ for $\cos 2\theta > 0$, $H = 0$ for $\cos 2\theta < 0$. In (b) solid and dashed curves indicate respectively $\tilde{V}_z^{(1)} < 0$ and $\tilde{V}_z^{(1)} > 0$. The fine-scale structure apparent in the contours close to the separatrix is a computational artefact.*

in the quadrants $\pi/4 < \varphi < 3\pi/4$, $5\pi/4 < \varphi < 7\pi/4$ and those in the other two quadrants. In the former the streamlines are essentially circles centred on the X-line, as in the previous solution, whereas elsewhere the streamlines are convex towards the X-line: A falls off with distance from the X-line much more rapidly along the x-axis than it does along the y-axis. However, the streamlines in the different regions connect smoothly onto each other at the separatrix. Adopting the current-free X-point configuration as a simple model of the divertor region of a tokamak plasma, it is natural for us to identify the quadrant $\pi/4 < \varphi < 3\pi/4$ as the confined plasma region. The fact that A is finite but evanescent in the two adjoining quadrants (i.e. outside the region of confined plasma, in the tokamak divertor picture) can be interpreted as a manifestation of wave tunnelling through the separatrix. It is apparent from Fig. 1 (a) that the strongest flows occur in this region, close to the X-line.

Shown in Fig 1 (b) are the contours of $\tilde{V}_z^{(1)}$, which as noted previously, can be regarded as a measure of the fast wave amplitude since it is identically zero for a shear wave propagating in the z-direction. Solid (broken) curves representing positive (negative) values of this quantity. The absolute value of this quantity is strongly peaked in the vicinity of the separatrix, although it also increases with distance from the X-line and vanishes at this point. Thus, the shear wave is most strongly coupled to the fast wave on the separatrix far from the X-line. Elsewhere, the $\tilde{V}_z^{(1)}$ eigenfunction has a rather complex structure, with O-points in the quadrants $\pi/4 < \varphi < 3\pi/4$, $5\pi/4 < \varphi < 7\pi/4$ and secondary X-points in the other two quadrants.

4.   Summary and conclusions

We have presented a linear analysis of Alfvén waves propagating along the X-line of a magnetic X-point configuration with a strong longitudinal field, in the limit of vanishing plasma beta and equilibrium plasma current. This scenario provides a simplified description of Alfvén wave propagation in the divertor region of tokamak plasmas. The presence of the X-point means that the structure of the leading order (shear Alfvén) eigenfunction is not arbitrary: we have obtained a general solution for the stream function A describing the leading order flow, and evaluated explicitly both this solution and fast wave corrections to it for two illustrative cases. In the first of these A is azimuthally symmetric in the X-point plane and singular at the X-line, while in the second case A is finite everywhere and largely confined to two quadrants in the X-point plane, one of which can be regarded as representing the confined plasma region in a tokamak. For this second scenario we have shown that strong coupling of the shear and fast waves occurs close to the separatrix far from the X-line. Bulanov and co-workers [4] also found that the separatrix plays a key role in the coupling of the two modes.

The singularity at the X-line in the case of the azimuthally symmetric solution for the leading order flow ($A = 1/r$) is of course unphysical, but it is very likely that it could be removed by extending the model to include non-ideal MHD effects, such as the Hall term in the induction equation. The $A = 1/r$ solution could have applications to the X-points of magnetic islands. It may be significant in this context that high frequency ($\sim 400 - 500\mathrm{kHz}$) density fluctuations, possibly indicative of fast wave activity, have been detected using pulsed radar reflectometry close to the X-points of islands in the TEXTOR tokamak [7]. The solution with non-azimuthally symmetric A, on the other hand, is more likely to have applications to divertor X-points; for example, it may help explain the formation of filamentary structures propagating away from the separatrix in MAST low and high confinement regimes [8].

References

[1] Bulanov, S. V., Shasharina, S. G. and Pegoraro, F. 1990 *PPCF* **32**, 377.

[2] McClements, *et al.* 2004 *Astrophys. J.* **609**, 423.

[3] Bulanov, S. V. and Syrovatskii, S. I. 1980 *Sov. J. Plasma Phys.* **6**, 661.

[4] Bulanov, S. V., Shasharina, S. G. and Pegoraro, F. 1992 *PPCF* **34**, 33.

[5] McClements, K. G., Shah, N. and Thyagaraja, A. 2006 *J. Plasma Phys.* **72**, 571.

[6] Ben Ayed, N., McClements, K. G., and Thyagaraja, A. 2008 *JPP* to be published.

[7] Donné, A. H., *et al.*, 2005 *Phys. Rev. Lett.* **94**, 085001.

[8] Kirk, A., *et al.* 2006 *PPCF* **48**, B433