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Intermittency in self-organised shear flows

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Introduction

Understanding multi-scale interactions is an outstanding problem in plasmas. Despite complex nonlinear dynamics, coherent structures such as shear flows often form from small-scale turbulence, which then feed back on small-scales. A remarkable consequence of this mutual interaction is self-organisation, which provides a powerful paradigm for understanding complexity in many systems (e.g. population, forest fires, reaction-diffusion). In this paper, we present a non-perturbative statistical theory of self-organisation of a shear flow, which can perhaps be utilized as an exploratory model in different contexts.

Statistical theory of self-organised shear flows

We consider a forced shear flow, whose gradient grows until it becomes unstable according to the stability criterion. For instance, in a strongly stably stratified medium, fluctuations on small scales (or internal gravity waves) will sharpen the structure of a shear flow u [1, 2], acting as a forcing, until its gradient $\partial_x u = u_x$ exceeds the critical value u_{xc}, set by Richardson criterion $R = R_c = (\mathcal{N}/u_{xc})^2 = 1/4$. Here $\mathcal{N}$ is the buoyancy frequency due to the restoring force (buoyancy) in a stably stratified medium. Once the flow is unstable, the shear flow will relax its gradient rapidly due to the onset of turbulence (fluctuations) until it starts building up again at the expense of fluctuations. In magnetically confined plasmas, poloidal shear flows (zonal flows) and/or parallel flows can be generated from drift waves while becoming subject to Kelvin-Helmholtz type instabilities. We model the dynamics of the shear flow by the following one dimensional (1D) nonlinear diffusion equation for u_x [3],

$$\partial_t u_x = \partial_{xx}[D(u)u_x] + f\,, \tag{1}$$

where

$$D(u) = \nu + \beta u_x^2\,. \tag{2}$$

In Eqs. (1)-(2), f is an external forcing; $D(u)$ represents the effective diffusion coefficient including both the molecular diffusivity ν and nonlinear (eddy) diffusivity capturing relaxation process for unstable shear flow $|u_x| > u_{xc}$. A similar quadratic eddy diffusivity has widely been used in modelling chemical mixing and angular momentum transport (e.g. in stars and the Sun)

although the precise value of parameter β has been controversial, often adjusted in an attempt to reproduce observational data [4].

Since for small $|u_x| \ll u_{xc}$, the forcing is balanced by linear diffusion, naturally leading to the Gaussian distribution of u_x, we focus on the PDF tails for large value of $|u_x|$ where the cubic nonlinearity becomes important. In order to incorporate this nonlinear interaction non-perturbatively, our key idea is to look for a nonlinear structure that is likely to be naturally sustained in a system. One candidate for such a nonlinear structure is an exact nonlinear solution $u_x \propto x$ to Eqs. (1)-(2) in the absence of the forcing. Due to a stochastic forcing, this structure is then likely to form in a random fashion with the temporal behaviour governed by $Q(t)$ as $u_x \sim iQ(t)x$. The PDF of u_x then becomes equivalent to that of $Q(t)$, which satisfies:

$$\partial_t Q = -\beta Q^3 + g\,, \tag{3}$$

where g is the time dependent part of the forcing with the spatial profile $\propto x$. In the case of a temporally short-correlated forcing

$$\langle g(t_1)g(t_2)\rangle = \delta(t_1 - t_2)G\,, \tag{4}$$

the Fokker-Planck equation for the PDF of Q can be derived by using a standard technique [6]. as

$$\partial_t P(Q,t) = \beta\,\partial_Q[Q^3 P] + G\partial_{QQ}P\,. \tag{5}$$

A stationary solution of Eq. (5) can easily be found to be $P(Q,t) \sim P_0 \exp[-\beta Q^4/(4G)]$, leading to

$$P(u_x;x,t) \sim P_0 \exp[-\beta u_x^4/4G]\,. \tag{6}$$

The PDFs tails in Eq. (6) are non-Gaussian, intermittent with the exponential scaling of $P \sim \exp(-\beta u_x^4/4G)$, and symmetric under the reflection $x \to -x$ (unlike Burgers turbulence [5] which is asymmetric). This exponential tail is one manifestation of intermittency caused by a coherent structure. Note that a similar exponential PDF tails can also be obtained by using the instanton method [7].

We test our analytical prediction (6) by performing direct numerical simulations of Eqs. (1) and (2). We use the Gaussian forcing f in Eq. (1), which is homogeneous, and temporally short-correlated, with the power spectrum $F(k)$:

$$\langle f(k_1,t_1)f(k_2,t_2)\rangle = \delta(t_1 - t_2)\delta(k_1 + k_2)F(k)\,. \tag{7}$$

The PDFs of u_x, $P(u_x)$, for a white-noise $F(k) = k^0$ are shown by the solid line in Fig. 1 for the values of parameters $\nu = 6 \times 10^{-3}$ and $\beta = 6.25 \times 10^{-3}$. It can clearly be seen that the

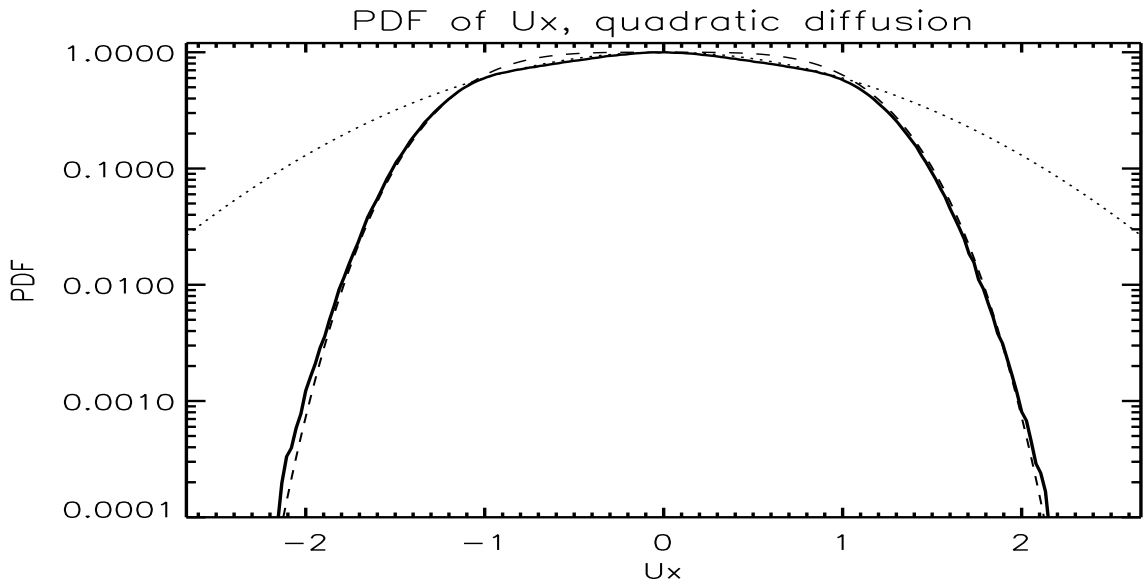

Figure 1: The solid line is the PDFs from the numerical simulation of a nonlinear diffusion model (2) for a white noise. The dotted and dashed lines are the fits to Gaussian and $\exp(-cu_x^4)$ (c =const).

PDF is Gaussian near the center but becomes exponential $\exp(-cu_x^4)$ in the tails (c = constant). These exponential tails agree perfectly with our theoretical prediction (6). To highlight this, the dotted and dashed lines in Fig. 1 are fits to a Gaussian and to $\exp(-cu_x^4)$, respectively. The cross-over between these two regimes occurs approximately at the expected critical gradient of $u_{xc} \simeq \sqrt{\nu/\beta} = 0.98$. The mean value of $|u_x|$ is found to be smaller than this, with the value about 0.59. However, there is yet a significant probability of 20% of super-critical gradient $|u_x| > |u_{xc}|$ from the PDF tails.

Mathematically, the PDF tails $\exp(-cu_x^4)$ result from the highest cubic nonlinearity in the equation for u_x (1), as shown by [8]. Physically, they are due to the feedback of shear on turbulence when it becomes unstable. That is, while shear is generated by turbulence [modelled by the forcing f in Eq. (1)], it feeds back on turbulence, limiting its own growth, thereby reducing the PDF tails below the Gaussian prediction (see Fig. 1). We have confirmed that these exponential PDFs tails are robust features by using different power spectra $F(k) \sim k^{-1}$ and k^{-2}. However, note that relative time scales of relaxation and disturbance are crucial in determining the validity of the nonlinear diffusion model [7].

Conclusion

We have presented a statistical theory of self-organisation by utilizing a simplified nonlinear diffusion model for a shear flow and a widely invoked quadratic eddy diffusivity [3, 4]. Both analytical and numerical simulations predict the PDF tails of the exponential form $\exp(-cu_x^4)$,

with a strong intermittency. It is very interesting that exponential scalings have often been observed in the tails of fluxes in laboratory plasmas (e.g. see Refs. [9, 10]). Our numerical simulation also reveal that the significant contribution from the PDF tails with a large population of super-critical gradients, which could play a crucial role. These results highlight the importance of the statistical description of gradients in self-organisation, rather than its average value as has conventionally been done. These results can have important implications for the dynamics and the role of shear flows (e.g. zonal flows) in laboratory, astrophysical and geophysical plasmas, which is vital not only in momentum transport, but also in transporting chemical species and controling mixing of other quantities (e.g. air pollution, weather control) [11, 12].

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