Finite-banana-width Effects on Stability Threshold of NTMs: Formalism

M. Reshko[1], H. R. Wilson[1], H. Qu[2]

1 Department of Physics, University of York, Heslington, York, YO10 5DD, UK

2 Centre for Fusion Science, SWIP, Chengdu, Sichuan, 10041, China

Introduction

Determination of the threshold island width for stability of the neoclassical tearing mode in tokamaks is important for understanding the conditions for its growth, and for developing experimental control techniques. The polarisation current may provide such a threshold mechanism when the width of the island, *w*, is comparable to the trapped ion banana orbit width, ρ_{Bi}. We present a kinetic formalism for investigating the impact of the perturbed bootstrap and neoclassical ion polarisation currents on stability of NTMs in tokamaks while retaining full finite-ion-banana-width effects, focusing on small islands with $w \sim \rho_{Bi}$.

The Model

We consider a single dominant helicity perturbation propagating in the toroidal direction with frequency ω such that in a stationary laboratory frame of reference the total magnetic field is given by $\mathbf{B}_{\text{lab}} = I\nabla\phi + \nabla\phi\times\nabla\chi + \nabla\phi\times\nabla\psi$, where $\psi = \tilde{\psi}\cos\left(m\theta - n(\phi-\omega t)\right)$ and (χ,θ,ϕ) are partially orthogonal flux coordinates with $\nabla\phi\cdot\nabla\chi = \nabla\phi\cdot\nabla\theta = 0$. The island is centred on a rational surface $\chi=\chi_c$ with $q(\chi_c) = q_c = m/n$, where $q = (\mathbf{B}\cdot\nabla\phi)/(\mathbf{B}\cdot\nabla\theta) = JIR^{-2}$ and $J^{-1} = \nabla\chi\times\nabla\theta\cdot\nabla\phi = \mathbf{B}_{\text{lab}}\cdot\nabla\theta$ is the Jacobian, and has a half-width *w*, which in units of magnetic flux is given by $w_\chi^2 = 4\tilde{\psi}\, q_c/q_c'$, where $q_c' = (dq/d\chi)_{\chi=\chi_c}$.

We assume that the time variation of the system is due solely to the propagation of the island and hence can be eliminated by transforming into its rest frame with coordinates (χ,θ,ζ), where $\zeta = \phi - \omega t$. In this rotating reference frame the drift-kinetic equation for the gyro-phase independent part of the distribution function f_s for plasma species *s* is given by

$$\left(\mathbf{v}_\parallel + \mathbf{v}_D\right)\cdot\nabla f_s + \left(\mathbf{v}_\parallel + \mathbf{v}_D\right)\cdot\left(m_s\boldsymbol{\omega}\times\mathbf{V} - q_s\nabla\Phi_{\text{rot}}\right)\frac{\partial f_s}{\partial\varepsilon} = C(f_s), \tag{1}$$

where $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ with $\hat{\mathbf{z}}$ being the unit vector in the direction of toroidal axis, $\mathbf{V} = \omega R^2\nabla\phi$,

$\mathbf{v}_D = \dfrac{\mathbf{b}\times\nabla\Phi_{\text{rot}}}{B_{\text{rot}}} - \mathbf{v}_\parallel\times\nabla\dfrac{v_\parallel}{\omega_{cs}} - \dfrac{1}{\omega_{cs}}\mathbf{b}\times\left(\boldsymbol{\omega}\times\left(2\mathbf{v}_\parallel + \mathbf{V}\right)\right)$, $\varepsilon = \dfrac{m_s v^2}{2}$, $\omega_{cs} = \dfrac{q_s B_{\text{rot}}}{m_s}$, q_s and m_s are the electric charge and mass of the species *s*, and *C* is a collision operator. In the non-relativistic limit, i.e. neglecting terms of order $V/c \ll 1$ where *c* is the speed of light, the electro-magnetic fields in the rotating and laboratory frames are related via $\mathbf{B}_{\mathrm{rot}} = \mathbf{B}_{\mathrm{lab}}$ and $\mathbf{E}_{\mathrm{rot}} = \mathbf{E}_{\mathrm{lab}} + \mathbf{V} \times \mathbf{B}_{\mathrm{lab}}$. Since we will be working exclusively in the rotating frame, we drop the subscripts on the electro-magnetic fields from now on. We will make use of two coordinate systems: (χ, θ, ξ) with $\xi = m\theta - n\zeta$, and (Ω, θ, ξ) with $\Omega = 2(\chi - \chi_c)^2 / w_\chi^2 - \cos\xi$ where the parallel derivative operator near the island is given by

$$\mathbf{b} \cdot \nabla = \frac{1}{JB} \frac{\partial}{\partial\theta}\bigg|_{\chi,\xi} - m(\chi - \chi_c) \frac{q_c'}{q_c} \frac{1}{JB} \frac{\partial}{\partial\xi}\bigg|_{\chi,\theta} + m\tilde{\psi} \sin\xi \frac{1}{JB} \frac{\partial}{\partial\chi}\bigg|_{\theta,\xi},$$

$$\mathbf{b} \cdot \nabla = \frac{1}{JB} \frac{\partial}{\partial\theta}\bigg|_{\Omega,\xi} - m(\chi - \chi_c) \frac{q_c'}{q_c} \frac{1}{JB} \frac{\partial}{\partial\xi}\bigg|_{\Omega,\theta}$$

respectively, where **b** = **B** / B. We express the distribution function as

$$f_s = \left(1 - \frac{q_s \Phi_1}{T_s}\right) F_{Ms} + g_s,$$

where $F_{Ms}(\chi, w)$ is the Maxwellian, and define a small parameter $\Delta = w / r_c \ll 1$, where r_c is the minor radius of the surface with $q = m/n$, and adopt the following orderings

$$\frac{w}{F_{Ms}} \frac{\partial F_{Ms}}{\partial r} \sim \frac{q_s \Phi_1}{T_s} \sim \frac{g_s}{F_{Ms}} \sim \Delta \quad \text{and} \quad \frac{w}{F_{Ms}} \frac{\partial g_s}{\partial r} \sim \frac{q_s \Phi_0}{T_s} \sim \frac{\omega}{\omega_{*i}} \sim 1,$$

where $\omega_{*i} = \frac{T_i}{q_i} \frac{1}{n_i} \frac{dn_i}{d\chi}$ is the ion diamagnetic frequency.

Electron Response

In addition to the assumptions set out in the preceding section, we further define a second small parameter for electrons $\delta = \rho_{Be} / w \ll 1$, where ρ_{Be} is the width of the electrons' banana orbits, and perform a double expansion of g_e: $g_e = \sum_j \sum_k \delta^j \Delta^k g_e^{(j,k)}$.

The details of the calculation may be found in [1], while here we just present the results:

$$g_e^{(0,1)} = \left(\chi - \chi_c + h_e^{(0,1)}(\Omega)\right) \left(\frac{\partial F_{Me}}{\partial\chi}\bigg|_{\theta,\xi} + \frac{q_e F_{Me}}{T_e} \frac{d\Phi_0}{d\chi}\bigg|_{\theta,\xi} \right)\Bigg|_{\chi=\chi_c}, \qquad (2)$$

and

$$g_e^{(1,1)} = -\frac{\partial}{\partial \theta}\left(\frac{I v_\parallel}{\omega_{ce}}\right)\Bigg|_{\chi,\xi}\left(\frac{\partial F_{Me}}{\partial \chi}\Bigg|_{\theta,\xi} + \frac{q_e F_{Me}}{T_e}\frac{d\Phi_0}{d\chi}\Bigg|_{\theta,\xi} + \frac{\partial g_e^{(0,1)}}{\partial \chi}\Bigg|_{\theta,\xi}\right) + h_e^{(1,1)}, \tag{3}$$

where the free function $h_e^{(0,1)}(\Omega)$ arising from integration along a field line is determined from considerations of radial transport.

Ion Response

For ions we assume that the width of their banana orbits is comparable to the half-width of the magnetic island and expand g_i as $g_i = \sum_j \Delta^j g_i^{(j)}$. Thus, at order $O(\Delta^1)$ the drift-kinetic equation (1) for ions is given by

$$\frac{v_\parallel}{JB}\frac{\partial g_i^{(1)}}{\partial \theta}\Bigg|_{\chi,\xi} + \frac{v_\parallel}{JB}\frac{\partial}{\partial \theta}\left(\frac{I v_\parallel}{\omega_{ci}}\right)\Bigg|_{\chi,\xi}\frac{\partial g_i^{(1)}}{\partial \chi}\Bigg|_{\theta,\xi} = -\frac{v_\parallel}{JB}\frac{\partial}{\partial \theta}\left(\frac{I v_\parallel}{\omega_{ci}}\right)\Bigg|_{\chi,\xi}\left(\frac{\partial F_{Mi}}{\partial \chi}\Bigg|_{\theta,\xi} + \frac{q_i F_{Mi}}{T_i}\frac{d\Phi_0}{d\chi}\Bigg|_{\theta,\xi}\right). \tag{4}$$

Defining $p = \chi - I v_\parallel / \omega_{ci}$, we note that

$$\frac{\partial}{\partial \theta}\Bigg|_{p,\xi} = \frac{\partial}{\partial \theta}\Bigg|_{\chi,\xi} + \frac{\partial}{\partial \theta}\left(\frac{I v_\parallel}{\omega_{ci}}\right)\Bigg|_{\chi,\xi}\frac{\partial}{\partial \chi}\Bigg|_{\theta,\xi} + O(\Delta)$$

and equation (4) can be expressed as

$$\frac{v_\parallel}{JB}\frac{\partial g_i^{(1)}}{\partial \theta}\Bigg|_{p,\xi} = -\frac{v_\parallel}{JB}\left(\frac{\partial F_{Mi}}{\partial \theta}\Bigg|_{p,\xi} + \frac{\partial G_i}{\partial \theta}\Bigg|_{p,\xi}\right),$$

where we have introduced a function $G_i = G_i(\chi,\varepsilon)$ such that $\frac{\partial G_i}{\partial \chi}\Big|_{\theta,\xi} = \frac{q_i F_{Mi}}{T_i}\frac{d\Phi_0}{d\chi}\Big|_{\theta,\xi}$. Thus,

$$g_i^{(1)} = H_i(p,\xi,\varepsilon,\lambda,\sigma) - F_{Mi} - G_i = h_i^{(1)} + (\chi - \chi_c)\left(\frac{\partial F_{Mi}}{\partial \chi}\Bigg|_{\theta,\xi} + \frac{q_i F_{Mi}}{T_i}\frac{d\Phi_0}{d\chi}\Bigg|_{\theta,\xi}\right)_{\chi=\chi_c} + O(\Delta^2) \tag{5}$$

where $h_i^{(1)} = H_i - F_{Mi}(\chi_c,\varepsilon) - G_i(\chi_c,\varepsilon) = h_i^{(1)}(p,\xi,\varepsilon,\lambda,\sigma)$.

At order $O(\Delta^2)$, after substituting the form (5) for $g_i^{(1)}$, expanding equilibrium quantities around the surface $\chi = \chi_c$ and keeping only the lowest order terms, transforming to (p,θ,ξ) coordinates and applying orbit-averaging operators defined as

$$\langle f \rangle = \left(\int_{-\pi}^{\pi}\frac{JB}{v_\parallel}d\theta\right)^{-1}\int_{-\pi}^{\pi} f\,\frac{JB}{v_\parallel}d\theta \quad \text{and} \quad \langle f \rangle = \left(\sum_\sigma \int_{-\theta_b}^{\theta_b}\frac{JB}{|v_\parallel|}d\theta\right)^{-1}\sum_\sigma \int_{-\theta_b}^{\theta_b} f\,\frac{JB}{|v_\parallel|}d\theta$$

for passing and trapped particles respectively with the integral performed at fixed $(p,\xi,\varepsilon,\lambda)$, we find that the drift-kinetic equation may be expressed in the following form

$$\left\langle \frac{I}{JB^2}\frac{\partial\Phi_1}{\partial\theta}\bigg|_{p,\xi} - \frac{I}{JB^2}\frac{\partial\hat{\rho}_{Bi}}{\partial\theta}\bigg|_{\chi,\xi}\frac{\partial\Phi_1}{\partial p}\bigg|_{\theta,\xi} + n\frac{\partial\Phi_1}{\partial\xi}\bigg|_{p,\theta}\right\rangle\left(\frac{\partial F_{Mi}}{\partial\chi}\bigg|_{\theta,\xi} + \frac{q_i F_{Mi}}{T_i}\frac{d\Phi_0}{d\chi}\bigg|_{\theta,\xi}\right)_{\chi=\chi_c}$$
$$= \left\langle C_1\left(h_i^{(1)}\right)\right\rangle - n\left(\left\langle\frac{\partial\Phi_1}{\partial p}\bigg|_{\theta,\xi}\right\rangle + \left(\frac{d\Phi_0}{d\chi}\bigg|_{\theta,\xi}\right)_{\chi=\chi_c} + pq_c'\frac{2\pi}{N}\right)\frac{\partial h_i^{(1)}}{\partial\xi}\bigg|_{p,\theta}$$
$$+\left(\left\langle n\frac{\partial\Phi_1}{\partial\xi}\bigg|_{p,\theta}\right\rangle - m\tilde{\psi}\sin\xi\frac{2\pi}{N} + \left\langle 2v_{\parallel}\frac{\omega}{\omega_{ci}}\hat{\mathbf{z}}\cdot\nabla\chi\right\rangle\right)\frac{\partial h_i^{(1)}}{\partial p}\bigg|_{\theta,\xi} \tag{6}$$
$$+n\left\langle\left(v_{\parallel}B\frac{\partial}{\partial\chi}\left(\frac{v_{\parallel}}{\omega_{ci}}\right)\bigg|_{\theta,\xi} + (\nabla\theta\cdot\nabla\chi)\frac{v_{\parallel}}{BR^2}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\omega_{ci}}\right)\bigg|_{\chi,\xi} + q_c'\frac{Iv_{\parallel}}{\omega_{ci}}\frac{v_{\parallel}}{JB}\right)_{\chi=\chi_c}\right\rangle\frac{\partial h_i^{(1)}}{\partial\xi}\bigg|_{p,\theta}$$

for passing particles and

$$\left\langle \frac{I}{JB^2}\frac{\partial\Phi_1}{\partial\theta}\bigg|_{p,\xi} - \frac{I}{JB^2}\frac{\partial}{\partial\theta}\left(\frac{Iv_{\parallel}}{\omega_{ci}}\right)\bigg|_{\chi,\xi}\frac{\partial\Phi_1}{\partial p}\bigg|_{\theta,\xi} + n\frac{\partial\Phi_1}{\partial\xi}\bigg|_{p,\theta}\right\rangle\left(\frac{\partial F_{Mi}}{\partial\chi}\bigg|_{\theta,\xi} + \frac{q_i F_{Mi}}{T_i}\frac{d\Phi_0}{d\chi}\bigg|_{\theta,\xi}\right)_{\chi=\chi_c}$$
$$= \left\langle C\left(h_i^{(1)}\right)\right\rangle + n\left\langle\left(v_{\parallel}B\frac{\partial}{\partial\chi}\left(\frac{v_{\parallel}}{\omega_{ci}}\right)\bigg|_{\theta,\xi} + (\nabla\theta\cdot\nabla\chi)\frac{v_{\parallel}}{BR^2}\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\omega_{ci}}\right)\bigg|_{\chi,\xi} + q_c'\frac{Iv_{\parallel}}{\omega_{ci}}\frac{v_{\parallel}}{JB}\right)_{\chi=\chi_c}\right\rangle\frac{\partial h_i^{(1)}}{\partial\xi}\bigg|_{p,\theta} \tag{7}$$
$$-n\left(\left\langle\frac{\partial\Phi_1}{\partial p}\bigg|_{\theta,\xi}\right\rangle + \left(\frac{d\Phi_0}{d\chi}\bigg|_{\theta,\xi}\right)_{\chi=\chi_c}\right)\frac{\partial h_i^{(1)}}{\partial\xi}\bigg|_{p,\theta} + \left(\left\langle n\frac{\partial\Phi_1}{\partial\xi}\bigg|_{p,\theta}\right\rangle + \left\langle 2v_{\parallel}\frac{\omega}{\omega_{ci}}\hat{\mathbf{z}}\cdot\nabla\chi\right\rangle\right)\frac{\partial h_i^{(1)}}{\partial p}\bigg|_{\theta,\xi}$$

for trapped particles. A benefit of this formalism is that it permits a non-perturbative treatment of finite banana width effects while reducing the dimensionality of the system by eliminating one of the spatial coordinates. This in turn significantly reduces the demand for computational resources thus allowing for the problem to be solved on a moderate computer cluster and a new kinetic code that uses this formalism is currently under development. This code will use an electric potential self-consistently determined through quasi-neutrality to accurately describe the polarisation current and the role it plays in the evolution of the island and, in particular, its impact on the stability threshold – an important issue for ITER.

Acknowledgment

This work was supported by the UK Engineering and Physical Research Council.

References

1. H. R. Wilson, J. W. Connor, R. J. Hastie, and C. C. Hegna, Phys. Plasmas 3, 248 (1996)