A new resolvent technique for calculating linear eigenspectra in tokamaks

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Introduction

The nonlinear, two-fluid, fully electromagnetic, global tokamak plasma turbulence code CUTIE [1] has consistently reproduced many features of experimental tokamak plasmas, see for example [2] and [3], but a full linear investigation of the code has never been undertaken. Simulations are presented here both using a new *linear* CUTIE initial value code to calculate the dominant linear mode, and introducing a new resolvent eigenvalue technique to find the entire linear spectrum of unstable, neutral and stable modes present in the same system.By starting with reduced physical systems it has been possible to benchmark the linear CUTIE code against existing simulations, beginning with resistive tearing modes using results from both Thyagaraja [4] and Militello [5]. The full linear eigenspectra and corresponding eigenfunctions of each mode is revealed for a variety of different conditions. As tearing modes have been extensively investigated, both in slab geometry [6] where growth rate can be shown to depend on resistivity as $\gamma \propto \eta^{3/5}$, and in cylindrical geometry, where a more complicated dispersion relation exists [5], they are particularly suitable for the purposes of benchmarking a numerical code as there are numerous analytical and numerical results available for comparison.

CUTIE physical model

We do not list the full equations of the nonlinear CUTIE code here and direct the readers to Refs. [1] and [3]. Instead presented is the first reduced system under consideration, namely visco-resistive MHD in which the vorticity and electromagnetic potentials are evolved. The system is based on a periodic cylinder model (r, θ, $\zeta = z/R$) with flux surfaces as concentric circles. All plasma properties are written as a sum of a flux surface averaged mean, e.g. $n_0(r,t)$, depending only on radius which in the linear model are fixed equilibria, and a fluctuating part δn depending on (r, θ, ζ, t). These fluctuations are Fourier expanded as $n_0 + \delta n = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\hat{n}_{m,n}(r,t)\exp(im\theta + in\zeta)$ where n_0 is the $m = n = 0$ Fourier component. Using these conventions and assuming no mean electric field and $n_0(r) = n_0$ the standard MHD equations for an incompressible plasma with isotropic resistivity and viscosity linearise to

$$\Theta = \rho_s^2\nabla_\perp^2\phi, \tag{1}$$

$$\frac{\partial\Theta}{\partial t} + V_A\nabla_{||}\rho_s^2\nabla_\perp^2\psi = V_A\rho_s\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{4\pi\rho_s}{cB_0}\frac{dj_o}{dr} + \nu\nabla_\perp^2\Theta, \tag{2}$$

$$\frac{\partial \psi}{\partial t} + V_A \nabla_{||} \phi = \frac{c^2 \eta}{4\pi} \nabla_{\perp}^2 \psi. \tag{3}$$

where Θ, ϕ and ψ are the fluctuating vorticity, electrostatic potential and magnetic potential (where the m,n suffix is dropped). The differential operators $\nabla_{||}$ and $\nabla_{\perp}$ act along and perpendicular to the toroidal magnetic field B_0, $\rho_s = V_{th}/\omega_{ci}$ is the ratio of the thermal velocity to the ion cyclotron frequency, V_A is the Alfvén velocity, and j_0 is the equilibrium current density in the kink term of Equation 2 which drives the tearing mode. Viscosity and resistivity appear as ν and η in Equations 2 and 3. It is worth noting that the use of the vorticity-electrostatic potential relation in Equation 1 prevents the appearance of a fourth order derivative in the viscous term of Equation 2, which makes the finite differencing of these equations more straightforward.

These PDEs are time evolved for a single value of m and n using the linear CUTIE code until the mode with the largest growth rate dominates the system. The same system is solved for single values of m and n using the resolvent eigenvalue technique which we describe in the following section.

The resolvent technique

While the initial value code evolves the linear CUTIE system in time until only the dominant linear mode is visible, the resolvent technique assumes $n_{m,n}(r,t) = \tilde{n}_{m,n}(r)\exp(-i\lambda t)$, where $\lambda \in \mathbb{C}$ so that $\frac{\partial}{\partial t} = -i\lambda$, and finds the entire spectrum of λ (we note an analogy here to the Laplace transform). To find λ by using the resolvent method we consider the inhomogeneous system

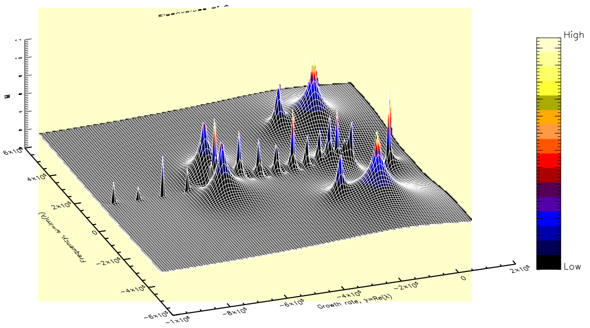

Figure 1: Poles of solution of inhomogeneous visco-resistive MHD system found by resolvent technique.

$$(L - \lambda^* I)\mathbf{x} = \mathbf{g} \tag{4}$$

where L is a linear operator derived from the PDEs with delta-function approximated $\mathbf{g}$ on the right hand side. Equation 4 is numerically solved for $\mathbf{x}$ as a function of the parameter λ^*, which is iteratively improved from an initial guess of a true eigenvalue λ. Using delta-function right hand sides is crucial since it helps ensure the right hand side vector is not orthogonal to any eigenvector, and it acts as a Greens function for the system. Starting from our guess λ^* we seek poles of the solution vector to the inhomogeneous system in the complex plane, see Figure 1, and at each iteration refine λ^* closer to the true eigenvalue λ. The technique is analogous to driving the system at a complex frequency λ^* and searching for resonances. Finally, as $\lambda^* \rightarrow \lambda$ the solution vector approaches an eigenvector corresponding to that eigenvalue. This method is a development of a similar idea used by Thyagaraja *et al* in [7], and is applicable to a wide range of systems whether the full system is self adjoint or not.

Results

The simulations presented here used to benchmark the linear CUTIE code and the resolvent CUTIE code were based on results from Thyagaraja [4] and Militello [5].

In comparison with Ref. [4] there are two main parameter spaces of interest, one a physically unrealistic regime in which the Lundqvist number $S = \tau_\eta/\tau_A = 2\times10^4$ and the other a tokamak-like regime with $S = 1\times10^7$. Here $\tau_A = a/V_A$ with $a = 100$cm, $R/a = 2.5$ and $V_A = 2.17\times10^8$cm s^{-1}. In both cases the q-profile was defined to be $q(r) = q(0)[1+(r/0.6a)^8]^{1/4}$ where $q(0) = 1.38$ with the mean current and resistivity profiles consistently calculated. For the former value of S cases corresponding to $M = \tau_v/\tau_A = 25, 400$ and 10^4 were considered for both the 2/1 and 3/2 modes. For the latter value of S values of M were chosen to be $10^5, 10^7$ and 10^9 and were considered only for the 2/1 modes. Low mode numbers are used in most cases since large wavelength modes are more unstable to linear tearing modes.

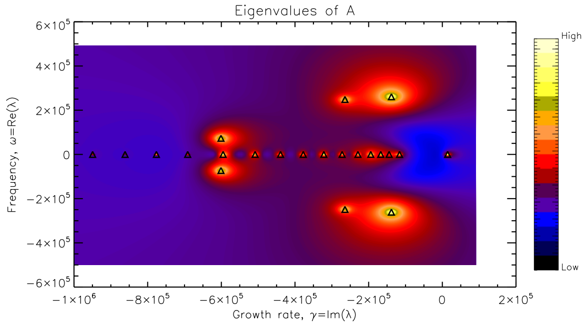

Figure 2: Linear eigenspectrum the visco-resistive MHD system with $S = \tau_\eta/\tau_A = 2\times10^4$ and $M = \tau_v/\tau_A = 400$. The complex plane measures growth rate and frequency in units of s^{-1}.

For all cases considered the growth rates and eigenfunctions from the linear CUTIE initial value code were an excellent match to the existing results. When simulated using the resolvent eigenvalue code the dominant modes were confirmed for all cases considered. In addition the stable part of the linear spectrum was found, and this can be seen in Figure 2 where the initial grid used to estimate the eigenvalue location in the complex plane is plotted as a contour, with the final calculated eigenvalues over-plotted for the case of $S = 2\times10^4$ and $M = 400$ for the 2/1 modes. The eigenfunctions of ψ for three distinct modes from these results are shown in Figure 3, corresponding to the tearing mode, an Alfvén wave close to marginal stability and a highly

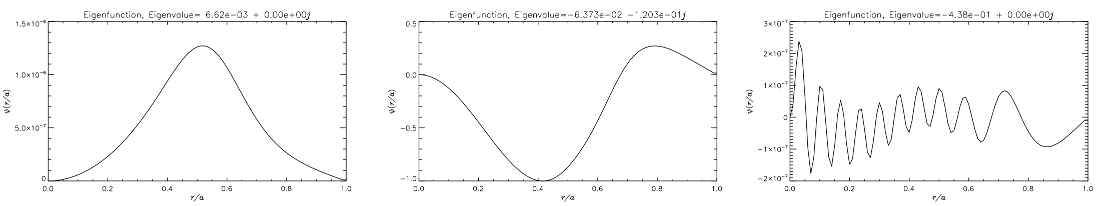

Figure 3: Eigenfunctions of ψ for unstable tearing mode (left panel), stable Alfvén wave (middle panel) and highly stable mode (right panel) in visco-resistive MHD system. The growth rates and frequencies have been normalised using τ_A.

stable mode exhibiting small scale structure.

The Militello eigenvalue code has been successfully tested against analytic results for tearing modes in cylindrical geometries [5]. For these benchmark simulations using the CUTIE system $q(r) = q(0)[1 + 16r^4]^{1/2}$ with $q(0) = 1.0$, $a = 25$cm and $R/a = 10$, with the equilibrium current density and resistivity profiles consistently calculated. Viscosity was held fixed at a negligible amount while resistivity was varied such that $\eta(0)$ was varied through several orders of magnitude, always considering the 2/1 mode.

The calculation of the growth rates of the dominant tearing mode is shown in Figure 4 and clearly shows excellent agreement to the Militello code from both the CUTIE initial value and resolvent codes. It is notable that good agreement is achieved both in the region of very low η in which the FKR [6] scaling of $\eta^{3/5}$ is recovered, and in the region of high η where the effects of cylindrical geometry alter this scaling [5]. Linear eigenspectra and corresponding eigenfunctions of all modes are also recovered from the resolvent code.

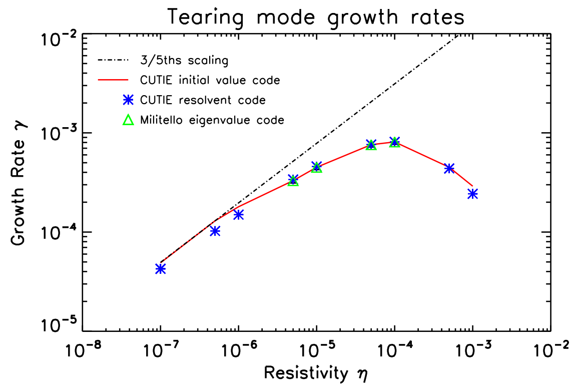

Figure 4: Growth rates of dominant tearing modes for different resistivities from the linear CUTIE code, the resolvent eigenvalue code, and the Militello eigenvalue code. A $\frac{3}{5}$ scaling over-plotted for comparison, and the growth rate is normalised using τ_A.

Conclusions

The resolvent eigenvalue code is a powerful technique for revealing the eigenspectra and the associated eigenfunctions of linear systems, including non self-adjoint systems and in principle higher dimensional systems involving coupled modes. This is a universal method which can be robustly applied to a variety of systems since it relies only on the ability to solve the inhomogeneous system. Here the method has been demonstrated to successfully find the full eigenspectra of visco-resistive MHD systems in a range of parameter spaces, and future work aims to shows its application beyond this including drift wave turbulence, ITG turbulence and curvature effects.

References

[1] A. Thyagaraja, Phys. Plasmas Cont. Fusion **42**, B255 (2000)
[2] A. Thyagaraja *et al*, Phys. Plasmas, **12**, 090907 (2005)
[3] A. Thyagaraja *et al*, Phys. Plasmas **17**, 042507 (2010)
[4] A. Thyagaraja, Phys. Plasmas Cont. Fusion, **36**, 1037 (1994)
[5] F. Militello *et al*, Phys. Plasmas, **11**, 125 (2003)
[6] H. Furth *et al*, Phys. Fluids, **6**, 459 (1963)
[7] A. Thyagaraja *et al*, J. Plasma Phys., **68**, 363 (2002)