Reynolds stress generation of poloidally asymmetric zonal flows and long-range correlations in fusion plasmas

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Today, it is widely accepted that transport barrier formation is mostly caused by a sheared radial electric field. The interplay between mean (i.e. zero-frequency, poloidally and toroidally symmetric) sheared flows and turbulence was modeled in a simple way in Ref. [1]. Actually, it was shown that several turbulence theories fitted the model, so it can be viewed as a paradigm for the understanding of mean sheared flow amplification by the Reynolds stress and turbulence suppression by shearing. In particular, the theory of resistive pressure-gradient-driven turbulence has been analyzed in detail from the point of view of such a paradigmatic model. The model consists of three ordinary differential equations for the level of turbulence, the mean flow shear and the pressure gradient.

Some experimental results, e.g. the detection of long-range potential correlations in TJ-II [2, 3], suggest that non-zero frequency, poloidally asymmetric zonal flows might play an important role in confinement transitions. The reason is that the observed long-range correlations are dominated by frequencies below 20 kHz and thus are not likely to be explained by toroidally and poloidally symmetric sheared flows. In Ref. [4] it was argued that long-range potential correlations might be an experimental evidence for the existence of non- zero frequency, poloidally asymmetric zonal flows and a phenomenological model within the aforementioned paradigm was formulated.

In this conference contribution we will try to investigate from a more fundamental perspective the effect of a zonal flow with poloidal mode number M=1 on the linear stability of resistive pressure-gradient-driven turbulence in cylindrical geometry. Specifically, our aim is to learn how the so-called g-modes are modified due to a zonal flow defined by a background electrostatic potential of the form

$$\Phi_0(r,\theta) = \Phi_{\rm av}(r) + \lambda\Phi_{\rm ZF}(r,\theta), \tag{1}$$

where both $\Phi_{\rm av}(r)$ and $\Phi_{\rm ZF}(r,\theta)$ have shear and λ controls the strength of the zonal flow. This requires to work out the linear solution for the electrostatic potential with a 'seed' zonal flow. Such a calculation is quite more complicated than for mean sheared flows. The reason is that for a seed mean sheared flow the linear eigenstate equation does not couple different poloidal and toroidal modes. However, if the seed sheared flow has a non-zero poloidal mode number, then the eigenstate equation ties together all poloidal modes and, strictly speaking, the poloidal mode number is not a good quantum number anymore. We will see how to obtain an approximate analytical solution to the problem. Once an appropriate expression for the linear eigenstate has been obtained, one can tackle the question of how the model proposed in Ref. [1] is modified and whether the phenomenological approach of Ref. [4] has some theoretical basis. The strategy consists in carrying out a quasilinear calculation of the Reynolds stress using the linear solutions. In this way, one can appropriately identify the structure of the Reynolds stress term and formulate a useful model for the interplay of poloidally symmetric and asymmetric sheared flows and the level of turbulence.

Resistive pressure-gradient-driven turbulence model

Our model consists of a reduced set of the resistive MHD equations (see Ref. [5]) formulated on a periodic cylinder with length $L = 2\pi R_0$. We use standard cylindrical coordinates (r,θ,z) so that the basis vectors are orthonormal and satisfy $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z$. The model in the electrostatic approximation and neglecting damping terms is:

$$\partial_t \nabla_\perp^2 \Phi + (\mathbf{v}\cdot\nabla)\nabla_\perp^2 \Phi = -\frac{B_0^2}{\rho\eta}\nabla_{||}^2\Phi - \frac{1}{\rho}\mathbf{e}_z\cdot(\nabla\Omega\times\nabla p), \qquad \partial_t p + (\mathbf{v}\cdot\nabla)p = 0, \quad (2)$$

where $\Omega(r)$ is the average helical curvature, ρ is the mass density and η is the resistivity; the last two quantities are assumed to be constant. Also, with $q(r) = \frac{r}{R_0}\frac{B_z}{B_\theta}$,

$$\mathbf{v} := -\nabla\Phi\times\mathbf{e}_z, \quad \nabla_{||}\Phi := \left(\frac{1}{R_0 q(r)}\partial_\theta + \partial_z\right)\Phi, \quad \nabla_\perp^2\Phi := \frac{1}{r}\partial_r(r\partial_r\Phi) + \frac{1}{r^2}\partial_\theta^2\Phi. \quad (3)$$

Linear eigenstates in the presence of zonal flows

Consider equilibrium quantities $\Phi_0(r,\theta)$ and $p_0(r)$. We want to study the linearization of the problem around this background:

$$\Phi(r,\theta,z,t) = \Phi_0(r,\theta) + \tilde{\Phi}(r,\theta,z,t), \quad p(r,\theta,z,t) = p_0(r) + \tilde{p}(r,\theta,z,t), \quad (4)$$

where $\Phi_0(r,\theta)$ has the form given in (1). It is obvious that a Fourier transform in z,

$$\tilde{\Phi}(r,\theta,z,t) = \sum_{n=-\infty}^{\infty}\tilde{\Phi}_n(r,\theta)e^{\gamma_n t - inz/R_0}, \quad \tilde{p}(r,\theta,z,t) = \sum_{n=-\infty}^{\infty}\tilde{p}_n(r,\theta)e^{\gamma_n t - inz/R_0}\,, \quad (5)$$

decouples the equations for different values of the toroidal mode number, n:

$$\gamma_n \nabla_\perp^2 \tilde{\Phi}_n + \frac{1}{r}\left[\partial_r \Phi_{\rm av}\partial_\theta + \lambda\left(\partial_r \Phi_{\rm ZF}\partial_\theta - \partial_\theta \Phi_{\rm ZF}\partial_r\right)\right]\nabla_\perp^2 \tilde{\Phi}_n$$

$$= -\frac{B_0^2}{\rho\eta R_0^2}\left(\frac{1}{q(r)}\partial_\theta - in\right)^2 \tilde{\Phi}_n - \frac{\Omega'(r)}{\rho r}\partial_\theta \tilde{p}_n,$$

$$\gamma_n \tilde{p}_n + \frac{1}{r}\left[\partial_r \Phi_{\rm av}\partial_\theta + \lambda\left(\partial_r \Phi_{\rm ZF}\partial_\theta - \partial_\theta \Phi_{\rm ZF}\partial_r\right)\right]\tilde{p}_n - \frac{1}{r}\partial_\theta \tilde{\Phi}_n p_0'(r) = 0. \quad (6)$$

However, for $\lambda \neq 0$, the Fourier transform in θ couples different poloidal mode numbers, m, due to the θ-dependence of the background electrostatic potential. In order to find an approximate solution we assume that the solution of (6) can be expressed as a power series in λ:

$$\tilde{\Phi}_n(r,\theta) = \sum_{j=0}^{\infty} \lambda^j \tilde{\Phi}_n^{(j)}(r,\theta), \quad \tilde{p}_n(r,\theta) = \sum_{j=0}^{\infty} \lambda^j \tilde{p}_n^{(j)}(r,\theta), \quad \gamma_n = \sum_{j=0}^{\infty} \lambda^j \gamma_n^{(j)}. \quad (7)$$

The strategy is as follows. We assume that the desired approximate solution is a perturbation of the zeroth order one, so that $\tilde{\Phi}_n \approx \tilde{\Phi}_n^{(0)} + \lambda \tilde{\Phi}_n^{(1)}$. The zeroth order equation is diagonal in the poloidal mode number and a solution $\tilde{\Phi}_{m,n}^{(0)}(r)$, $\tilde{p}_{m,n}^{(0)}(r)$, $\gamma_{m,n}^{(0)}$, labeled by m and n, can be found [6]. Now, the equation for $\tilde{\Phi}_{m,n}^{(1)}$ involves $\tilde{\Phi}_{m-1,n}^{(0)}$ and $\tilde{\Phi}_{m+1,n}^{(0)}$. Our aim is to obtain a satisfactory approximate solution to the first-order equations near a rational surface defined by $r_{M,N} = q^{-1}(M/N)$. The key point is to use that each $\tilde{\Phi}_{m,n}^{(0)}$ is very localized around $r_{m,n}$. The calculation is a bit cumbersome and we only quote here the final result, that can be expressed in terms of the width of the zeroth-order eigenfunction, W_{mn}, the average flow shearing rate $\omega_{mn} = V' m W_{mn}/r_{mn}$, the zonal flow shearing rate $\omega_{ZF,mn} = V'_{ZF} m W_{mn}/r_{mn}$, and the zeroth-order growth rate $\gamma_{mn}^{(0)}$. V' and V'_{ZF} denote the average and zonal flow shear amplitudes. Our approximate analytical solution reads:

$$\begin{aligned}
\tilde{\Phi}_{m,n}(r) &= \hat{\Phi}_{m,n}\exp\left(-\frac{(r-r_{m,n}+i\xi_{m,n})^2}{2W_{m,n}^2}\right) \\
&+ \hat{\Phi}_{m+1,n}(a_{m,n}+b_{m,n}(r-r_{m+1,n}))\exp\left(-\frac{(r-r_{m+1,n}+i\xi_{m+1,n})^2}{2W_{m+1,n}^2}\right) \\
&+ \hat{\Phi}_{m-1,n}(a^*_{-m,-n}+b^*_{-m,-n}(r-r_{m-1,n}))\exp\left(-\frac{(r-r_{m-1,n}+i\xi_{m-1,n})^2}{2W_{m-1,n}^2}\right). \\
&\equiv \tilde{\Phi}_{m,n}^{(0)}(r) + \tilde{\Phi}_{m,n}^{>}(r) + \tilde{\Phi}_{m,n}^{<}(r), \quad (8)
\end{aligned}$$

where $\xi_{m,n}/W_{m,n} \approx \omega_{m,n}/\gamma_{m,n}$ and in the last line we have introduced a notation that will be useful for writing the contributions to the Reynolds stress. A good approximation for the coefficients $a_{m,n}$ and $b_{m,n}$ is

$$a_{m,n} = -\frac{\omega^*_{{\rm ZF},m,n}}{\gamma_{m,n}^{(0)}}\frac{\omega_{m,n}}{\gamma_{m,n}^{(0)}}\frac{(nW_{m,n}q'(r_{m,n}))^4}{(nW_{m,n}q'(r_{m,n}))^2+1/2}, \quad b_{m,n} = -i\frac{\omega^*_{{\rm ZF},m,n}}{\gamma_{m,n}^{(0)}}\frac{1}{W_{m,n}}\frac{(nW_{m,n}q'(r_{m,n}))^2}{(nW_{m,n}q'(r_{m,n}))^2+1/2}.$$

Note that the zeroth-order contribution, $\tilde{\Phi}^{(0)}_{m,n}(r)$, is a tilted Gaussian localized around $r_{m,n}$ [6], whose tilt is proportional to the mean flow shear. The effect of the zonal flow shear is to generate non-negligible contributions for $\tilde{\Phi}_{m,n}(r)$ around the neighboring rational surfaces, namely $\tilde{\Phi}^{>}_{m,n}(r)$ and $\tilde{\Phi}^{<}_{m,n}(r)$.

Reynolds stress generation of zonal flows

A quasilinear estimation of the structure of the Reynolds stress shows how linear instabilities in the presence of $m = 1$ zonal flows can generate a non-vanishing $(\tilde{v}_r\tilde{v}_\theta)_{10}$ contribution. Schematically, around $r_{M,N}$,

$$\begin{aligned}(\tilde{v}_r\tilde{v}_\theta)_{10} &= -\frac{i}{r}\sum_{m,n} m\tilde{\Phi}_{m,n}\partial_r\tilde{\Phi}^{*}_{m-1,n} \\ &\approx -\frac{i}{r_{M,N}}\sum_{\substack{m,n\\ m/n=M/N}} m\tilde{\Phi}^{(0)}_{m,n}\partial_r\tilde{\Phi}^{>*}_{m-1,n} + (m+1)\tilde{\Phi}^{<}_{m+1,n}\partial_r\tilde{\Phi}^{(0)*}_{m,n} \\ &\quad - \frac{i}{r_{M,N}}\sum_{\substack{m,n\\ m/n=M/N}} m\tilde{\Phi}^{(0)}_{m,n}\partial_r\tilde{\Phi}^{(0)*}_{m-1,n} + (m+1)\tilde{\Phi}^{(0)}_{m+1,n}\partial_r\tilde{\Phi}^{(0)*}_{m,n} \end{aligned} \tag{9}$$

After some manipulations and including the necessary dissipation term one can write a model equation for the time evolution of the amplitude of the zonal flow shear, ${V_{\mathrm{ZF}}}'$,

$$\partial_t {V_{\mathrm{ZF}}}' = (a_1 - a_2 V'^2)E^2 {V_{\mathrm{ZF}}}' + cE^2 V' - b{V_{\mathrm{ZF}}}', \tag{10}$$

like the one proposed in Ref. [4].

Conclusions

We have found an analytical approximation to the linear eigenstates of resistive pressure-gradient-driven turbulence in the presence of a poloidally asymmetric sheared zonal flow. The poloidal asymmetry is responsible for non-negligible contributions to the $\tilde{\Phi}_{m,n}$ eigenfunction around neighboring rational surfaces, which in a quasilinear computation of the Reynolds stress produce new terms in agreement with those proposed in Ref. [4].

References

[1] P. H. Diamond, Y.-M. Liang, B. A. Carreras *et al.*, Phys. Rev. Lett. **72**, 2565 (1994).

[2] M. A. Pedrosa, C. Silva, C. Hidalgo *et al.*, Phys. Rev. Lett. **100**, 215003 (2008).

[3] C. Hidalgo, M. A. Pedrosa, C. Silva *et al.*, Europhys. Lett. **87**, 55002 (2009).

[4] I. Calvo, B. A. Carreras *et al.*, Plasma Phys. Control. Fusion **51**, 065007 (2009).

[5] B. A. Carreras *et al.*, Phys. Fluids **30**, 1388 (1987).

[6] B. A. Carreras *et al.*, Phys. Fluids B **5**, 1491 (1993).