Rankine-Hugoniot conditions and intermediate MHD shocks

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Abstract

The Rankine-Hugoniot jump conditions describe discontinuous solutions to the MHD conservation laws. Some of these connect a sub-Alfvénic to a super-Alfvénic region. These solutions to the Rankine-Hugoniot conditions are called intermediate shocks. We review and derive limiting shock parameter values for the existence of these intermediate shocks.

The isotropical sound speed is the characteristic speed for stationary hydrodynamics and the Prandtl-Meyer law states that every HD shock connects a subsonic to a supersonic state. In stationary magnetohydrodynamics, three highly anisotropical characteristic speeds exist. In increasing order: the slow magnetosonic speed, the Alfvén speed and the fast magnetosonic speed. A generalization to the Prandtl-Meyer law is thus not straightforward. It is well-known [4] that a shock cannot connect two states with the same relative speed magnitude compared to the characteristic speeds. Shocks connecting a subslow to a superslow state are called slow shocks, and shocks connecting a subfast to a superfast state are called fast shocks. All other shocks connect a sub-Alfvénic to a super-Alfvénic state and are called *intermediate shocks*. The existence of these intermediate shocks is still under debate (see [1] and references therein for the main arguments). Inspired by [3], we investigate under which parameter ranges the Rankine-Hugoniot conditions, which describe discontinuous solutions to the MHD equations, allow for intermediate shocks.

Rankine-Hugoniot conditions and their solution

We recalled in [1] that in any frame in which the shock is stationary (including the de Hoffmann-Teller frame [2]), the plasma is completely determined by three dimensionless parameters: the plasma-beta $\beta \equiv \frac{2p}{B_n^2+B_t^2}$, the inclination to the shock normal $\theta \equiv \frac{B_t}{B_n}$ and the Alfvén Mach number $M \equiv \sqrt{\frac{\rho v_n^2}{B_n^2+B_t^2}}$. Intermediate shocks are characterized by the fact that an $M > 1$ state connects to an unknown state which satisfies $M_u < 1$. Here we introduced mass density ρ, velocity $\mathbf{v}$, thermal pressure p and magnetic field $\mathbf{B}$. The index n refers to the direction of the shock normal, and the index t refers to the direction in the shock plane tangential to the shock normal. Let us now define the shock invariant

$$\xi \equiv \left((M^2-1)\theta, 2M^2+\beta(1+\theta^2)+\theta^2, (\frac{\gamma}{\gamma-1}\beta+M^2)(1+\theta^2)M^2\right). \tag{1}$$

The ratio of specific heats γ is a constant equation parameter. In [1], we showed that the RH conditions simplify as

$$[[\xi]] = 0, \tag{2}$$

and the primitive variables can be recovered from the dimensionless parameters. Here $[[\cdot]] \equiv \cdot_1 - \cdot_2$, where index 1 refers to the upstream state and index 2 refers to the downstream state.

Let now a known state $\mathbf{u}_k = (\beta,\theta,M)$ be given. We search the unknown states $\mathbf{u}_u = (\beta_u,\theta_u,M_u)$, which can be connected to $\mathbf{u}_k$ by a stationary MHD shock.

The solution is given by

$$M_u = \sqrt{\frac{(M^2-1)\theta+\theta_u}{\theta_u}}, \tag{3}$$

$$\beta_u = \frac{\left[(\gamma-1)((\theta-\theta_u)^2+(1+\theta^2)\beta)-4M^2\right](M^2-1)+2M^2(\theta+\theta_u)\theta_u}{(M^2-1)(\gamma+1)\left(1+{\theta_u}^2\right)}, \tag{4}$$

where θ_u can be found as the root of a cubic equation $C(\theta_u) = \Sigma_{i=0}^3 \tau_i \theta_u^i$ and τ_i are polynomial functions of (β,θ,M). It is well known that a cubic has three different real solutions if and only if

$$\Omega \equiv 27\tau_0^2 + 4\tau_1^3 + 4\tau_2^2\tau_0 - \tau_2^2\tau_1^2 - 18\tau_2\tau_1\tau_0 < 0. \tag{5}$$

Physical interpretation and visualisation of Ω

It is well-known [4] that when only one real root exists, it cannot lead to an intermediate shock solution. Therefore an intermediate shock can only exist when $\Omega < 0$. Moreover, in [1] we argue that the surface $\omega \equiv \{(\beta,\theta,M)|\Omega = 0\}$ in our 3-dimensional parameter state space represents exactly the states which can be connected to a state where v_n equals one of the characteristic speeds. Although this theory is generally applicable (for $\gamma > 1$), for the figures we assume that $\gamma = \frac{5}{3}$. The left panel of Fig. 1 shows these characteristic speeds in the $\beta = \frac{1}{10}$ cut of the (β,θ,M) parameter space. Also i-state regions are defined graphically. The right panel shows the surface ω, which separates the regions where the RH allow for three real solutions, from the regions where only one real solution exists.

Admissibility of solutions

There are three more physical restrictions for admissibility of MHD shocks, namely:

- $\rho_u > 0$. The conservation of momentum makes sure that this condition is trivially satisfied, when the normal velocities in both states have the same sign;

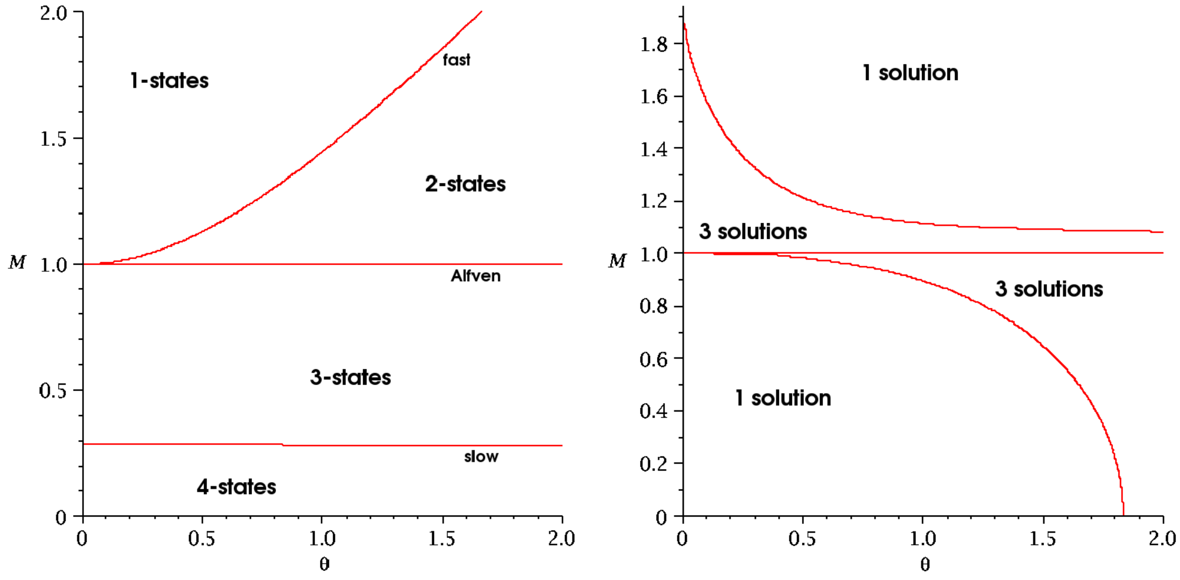

Figure 1: *Left:* The (θ, M) state plane for $\beta = \frac{1}{10}$ and $\gamma = \frac{5}{3}$. Shown are the curves fast: $v_n = v_f$, Alfvén: $v_n = a_n$ and slow: $v_n = v_s$. These curves separate the (θ, M) plane in the classical $1-2-3-4$ state regions. *Right:* Shown is the curve ω. Where $\Omega > 0$, only one real solution to the RH conditions exists. Where $\Omega < 0$, the RH conditions allow three real solutions.

- $p_u > 0$. This condition is non-trivial. The graphical representation of this requirement is shown in figure 2. In [1] we showed that if k of the solutions lead to negative p_u, it are exactly all the i-state solutions, with $i \leq k$;

- Entropy should increase. This condition can be satisfied by choosing the correct signs for v_n. When the upstream state is an i-state and the downstream a j-state, this condition is equivalent to $j > i$ (see e.g. [1] or [4]). In this case, the shock is called an $i \to j$ shock.

Regimes for intermediate shocks

We are now ready to formulate our results. Both frames of Fig. 2 show the $\beta = \frac{1}{10}$ cut of parameter space, where ω and the characteristic speeds are overplotted. Also, the critical surfaces for positive unknown pressure p_u are plotted. In the coloured regions, the RH conditions allow for intermediate shocks. The regions I, II and II allow for three different shock solutions, the regions IV and V allow for two shock solutions, the regions VI and VII allow for a single shock solution and region VIII only allows for negative pressure solutions. Not all of these solutions are of course intermediate shocks. In regions IV and V, the 1-state solution leads to negative p_u and in regions VI and VII the 1-state and 2-state solution lead to negative p_u. Therefore the right panel of figure 2 shows the regions where intermediate shock solutions are possible. In region I both a $1 \to 3$ and $1 \to 4$ solution exist, while in regions II and VI both a $2 \to 3$ and

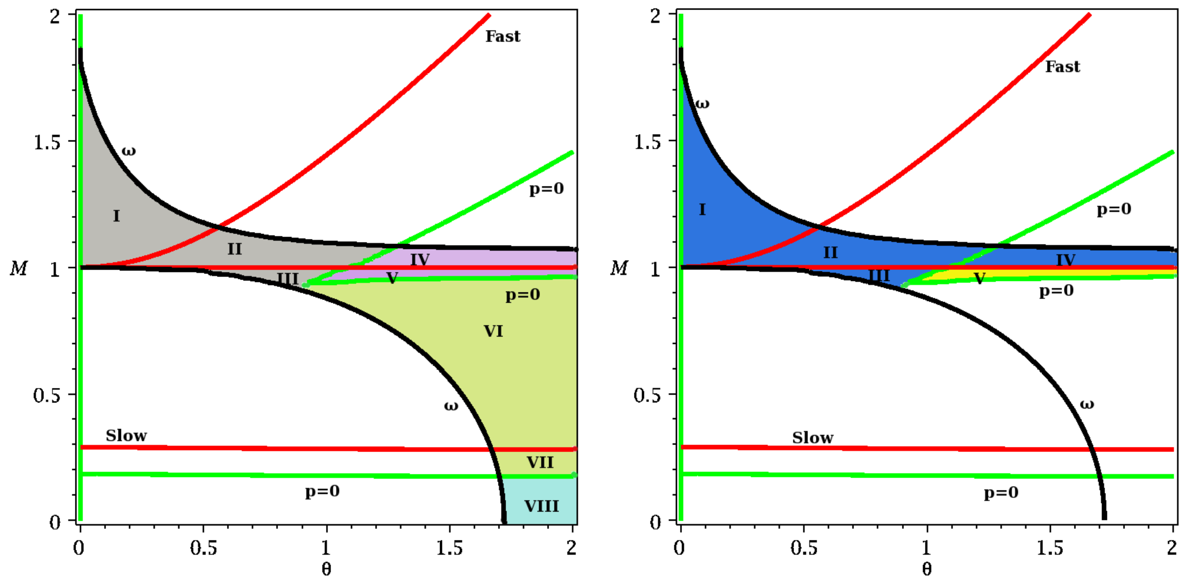

Figure 2: Left: Regions in which the RH conditions allow for intermediate shocks are coloured. Right: the blue regions allow for two different intermediate shocks. The yellow region allows for a single intermediate shock.

$2 \rightarrow 4$ intermediate shock solution exist. The difference between those two regions is that region II also leads to a fast shock solution. Also region III allows for two intermediate solutions: a $1 \rightarrow 3$ and a $2 \rightarrow 3$ solution. Region V finally only allows for intermediate $2 \rightarrow 3$ shocks.

Applied on $1 \rightarrow 3$ shocks

As a simple example we derive a limiting value for the downstream θ for $1 \rightarrow 3$ shocks. Therefore we need to fill out $M = 1$ in $p_u = 0$, and solve for θ. For fixed β, the solution is given by $\theta = \sqrt{\frac{(\gamma-1)\beta^2+(\gamma-3)\beta-2\sqrt{\beta(\gamma\beta+\gamma-1)}}{(\gamma-1)(\beta+1)^2}}$, for $\beta \in]0, \frac{4}{\gamma-1}]$. Note that the maximum value $\frac{1}{\sqrt{\gamma-1}}$ is reached for $\beta = \gamma^{-1}$. Therefore no $1 \rightarrow 3$ shocks are possible for $\theta > (\gamma-1)^{-1/2} = 0.77460$.

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