NUMERICAL CALCULATIONS FOR RELATIVISTIC ELECTRON CYCLOTRON DAMPING WITH AN ARBITRARY DISTRIBUTION FUNCTION AT ARBITRARY HARMONICS

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The relativistic expressions for the anti-hermitian (damping) parts of the dielectric tensor elements can be expressed as a single integral over the parallel momentum variable (see Eq.(1) below). A computer program has been written for the calculation of this single integral [1]. The numerical results are tested for a relativistic Maxwellian distribution and agree with analytical expressions derived for this case [1,2]. The computer code is therefore an essential element for the validation of powerful analytical results. The computer code is then applied to calculate electron cyclotron damping at arbitrary harmonics for any electron distribution function, distorted for example by an electric field or by both electron cyclotron and lower hybrid waves. For generality, we also include the case of relativistic Landau damping.

The anti-Hermitian relativistically-correct expressions for the dielectric tensor elements, ε_{ij}, are given for example in Ref. [3]. We choose the option of working with expressions involving a final integral only over the normalized parallel momentum, $z_{\|}$. To obtain damping, we require that $n_{\|}^2 - 1 + n^2\omega_c^2/\omega^2 > 0$ where $n_{\|}$ is the parallel refractive index of the medium. Since we are interested in damping close to a *n*-th cyclotron harmonic, $\omega \approx n\omega_c$ with $n \geq 1$, we focus our attention on a specific *n*. We have for the relativistic anti-Hermitian part:

$$\underline{\underline{\varepsilon}}^a = -i2\pi^2\left(\omega_p^2/\omega^2\right)\int dz_{\|} z_{\perp}^{res} \left|\underline{\underline{V}}^{res} U^{res}\right|_{z_{\perp}=z_{\perp}^{res}} \tag{1}$$

At resonance we have : $1 - n_{\|} z_{\|}/\gamma - n\omega_c/(\gamma\omega) = 0$. With $\gamma = \left(1 + z_{\|}^2 + z_{\perp}^2\right)^{1/2}$, we can rewrite this resonance condition as follows :`

$$z_{\perp}^2 + \left(z_{\|}(1-n_{\|}^2)^{1/2} - n_{\|}\frac{n\omega_c}{\omega}\frac{1}{(1-n_{\|}^2)^{1/2}}\right)^2 = \frac{n^2\omega_c^2/\omega^2 - 1 + n_{\|}^2}{1-n_{\|}^2} \tag{2}$$

which is the equation of an ellipse, and which can be rewritten along the resonance curve with the following parameters to be used in Eq.(1):

$$z_\perp^{res}(z_\parallel)=\left[(n_\parallel^2-1)z_\parallel^2+2n_\parallel z_\parallel\frac{n\omega_c}{\omega}-1+\left(\frac{n\omega_c}{\omega}\right)^2\right]^{1/2}\ ;\quad z_\parallel^\pm\equiv\frac{n_\parallel}{1-n_\parallel^2}\frac{n\omega_c}{\omega}\pm\frac{\left(n_\parallel^2-1+n^2\omega_c^2/\omega^2\right)^{1/2}}{\left|1-n_\parallel^2\right|}\ ;\ (3)$$

The limits of the integral are $z_\parallel^-$ to $z_\parallel^+$ if $1>|n_\parallel|$, or ∞ to $z_\parallel^-$ if $-1\geq n_\parallel$, or $z_\parallel^+$ to ∞ if $n_\parallel>1$.

Here: $z_{\parallel,\perp}\equiv p_{\parallel,\perp}/mc,\ n_{\parallel,\perp}\equiv k_{\parallel,\perp}c/\omega$, and $U^{res}=\left(\frac{n\omega_c}{\omega}\frac{\partial f}{\partial z_\perp}+n_\parallel z_\perp\frac{\partial f}{\partial z_\parallel}\right)^{res}$.

$V_{11}=V_1V_1$, $V_{21}=-V_{12}=V_2V_1$, $V_{22}=-V_2V_2$, $V_{13}=V_{31}=V_1V_3$, $V_{23}=-V_{32}=V_2V_3$, $V_{33}=V_3V_3$,

$V_1=nJ_n(x)/x,\quad V_2=idJ_n(x)/dx,\quad V_3=(z_\parallel/z_\perp)J_n(x),\quad x=n_\perp z_\perp\omega/\omega_c$, and

$f=f(z_\parallel,z_\perp)$ is the electron distribution, normalized so that $f=f_{actual}(mc)^3,\ 2\pi\int f\,d^2z=1$.

We first study the case of a relativistic Maxwellian distribution, namely

$$f(z_\perp,z_\parallel)=\{\mu/[4\pi K_2(\mu)]\}\exp\left[-\mu\left(1+z_\parallel^2+z_\parallel^2\right)^{1/2}\right].\qquad(4)$$

Here K is a modified Hankel functions, $v_t^2=T/m$ and $\mu\equiv(c/v_t)^2$. Damping occurs when $n_\parallel^2-1+n^2\omega_c^2/\omega^2>0$. Analytical expressions in this case are given in [1-3], when $1>|n_\parallel|$ and $n_\parallel^2>1$. For the limiting cases of $n_\parallel^2=0$ or $n_\parallel^2=1$, results are given in [1,2] for Landau damping.

Table 1 summarizes some results obtained near the fundamental (n=1) cyclotron frequency, the second (n=2) harmonic frequency, and for Landau damping (n=0). These results use the parameters:`ion charge Z=1, $T_e=4$ keV, $T_i=1$ keV, $n\omega_c/\omega$=0.95, $(n_\parallel)_{EC}$=0.6, $(n_\perp)_{EC}$= 0.4. For the case of a LH wave, $(\omega_c/\omega)_{LH}$ = 20., $(n_\parallel)_{LH}$=2.5 and $(n_\perp)_{LH}$=1.4. The results for a relativistic Maxwellian distribution calculated analytically from the results in [1-3], and numerically using Eq.(1), are first presented and show good agreement between the two sets of results. It is of course interesting to recover from Eq.(1) the same results as those obtained from the analytical expressions in [1-3], which involve complicated analysis with strings of Bessel functions. Note that the importance of relativistic correction in the calculation of Landau damping has been discussed in [4].

We next repeat the numerical calculations with the same parameters as in Table 1, for two distorted distribution functions calculated using the Fokker-Planck code presented in [5], time-step $\Delta t=0.1$, 400x200 grid points. Trapping is included, with the parameters of TdeV tokamak R_{min} =4.cm, R_{maj}=83.cm, maximum momentum P_{max}=30 (normalized to thermal momentum). We first treat the case of a runaway distribution function, calculated with an electric field normalized to the Dreicer field equal to $\varepsilon = 0.02$ .The differences of these results with those for a Maxwellian distribution are moderate for the cyclotron damping, but more substantial for the Landau damping. We finally repeat the numerical calculations for the case of a distribution function distorted by an EC wave plus a LH wave. We use a quasilinear LH diffusion coefficient D_{LH}=0.5 in the velocity range $\upsilon_1 \le p_{\|}/\gamma \le \upsilon_2$, ($\upsilon_1 = 3$, $\upsilon_2 = 6$), and D_{LH}=0 otherwise (velocity normalized to the thermal velocity, and momentum to the thermal momentum) , $\gamma^2 = 1 + z^2 = 1 + p^2\beta_{th}^2$, $\beta_{th} = \upsilon_{th}/c$ =0.04424 $(T_e(keV))^{1/2}$. The LH diffusion coefficient is constant along a plateau limited by $\upsilon_{1,2} = p_{\|}/\gamma$, which is rewritten:

$$p_{\|}^2(1-\upsilon_{1,2}^2\beta_{th}^2) - p_{\perp}^2\upsilon_{1,2}^2\beta_{th}^2 = \upsilon_{1,2}^2 \qquad (5)$$

Eq.(5) is the equation of two hyperbolae at the plateau boundaries $\upsilon_{\|} = \upsilon_1$ and $\upsilon_{\|} = \upsilon_2$ .For the EC wave, we use the quasilinear diffusion operator:

$$D_{EC} = D_{cy}\frac{\gamma}{|p_{\|}|}p_{\perp}^2 \exp\left(-\left(\frac{\gamma - n\omega_c/\omega}{p_{\|}\beta_{th}\Delta n_{\|}} - \frac{n_{\|}}{\Delta n_{\|}}\right)^2\right), \qquad (6)$$

Particles resonant with the wave are such that $n_{\|} = \left(\gamma - \frac{n\omega_c}{\omega}\right)/p_{\|}\beta_{th}$, which is rewritten:

$$p_{\perp}^2 + \left[p_{\|}(1-n_{\|}^2)^{1/2} - \frac{n\omega_c}{\omega}\frac{n_{\|}}{\beta_{th}}\frac{1}{(1-n_{\|}^2)^{1/2}}\right]^2 = \frac{1}{\beta_{th}^2}\frac{n^2\omega_c^2/\omega^2 - 1 + n_{\|}^2}{1-n_{\|}^2} \qquad (7)$$

Which is equivalent to Eq.(2). With D_{cy} = 0.03, $\Delta n_{\|}$ = 0.03, the resulting current obtained from the Fokker-Planck code [5] is J= 0.0877, the absorbed power $P = 5.923 \times 10^{-3}$, $J/P = 14.78$. The results in Table 1 indicate important deviations from the numerical results of a Maxwellian distribution for the harmonic n=2 and for Landau damping.

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Table 1 – Computed values for the anti-hermitian dielectric tensor elements

Assumed Values: $T_e = 4$ keV, $n\omega_c / \omega = 0.95$, $(n_\|)_{EC} = 0,.6$, $(n_\perp)_{EC} = 0.4$.

	Fundamental Damping ***n* = 1**		**Second Harmonic Damping** ***n* = 2**		**Landau Damping with $(n_\|)_{LH} = 2.5$** ***n* = 0**	
	Analytical	**Numerical**	**Analytical**	**Numerical**	**Analytical**	**Numerical**
					$a_{33} = 1$	
					$a_{32} = (v_{LH} / c)^2 (n_\|)_{LH} k_\perp c / \omega_c$	
Normal-ized to:	$\frac{\omega_p^2}{\omega^2}$	$\frac{\omega_p^2}{\omega^2}$	$10^{-2} \frac{\omega_p^2}{\omega^2}$	$10^{-2} \frac{\omega_p^2}{\omega^2}$	$a_{22} = 2\, a_{32}^2$ $10^{-2} \frac{\omega_p^2}{\omega^2} a_{ij}$	$10^{-2} \frac{\omega_p^2}{\omega^2} a_{ij}$
A relativistic Maxwellian distribution.						
ε_{11}^a / i	5.8255	5.8063	2.6721	2.6633		
ε_{12}^a	5.8255	5.7996	2.6721	2.6573		
ε_{22}^a / i	5.8255	5.7929	2.6721	2.6513	0.086554	0.086358
ε_{13}^a / i	0.28443	0.28367	0.14466	0.14412		
ε_{32}^a	0.28443	0.28331	0.14466	0.14376	-0.10304	-0.09309
ε_{33}^a / i	0.014233	0.014202	0.0080649	0.0080310	0.12180	0.12314
A runaway distribution, distorted by an electric field $\varepsilon = 0.02$. current = .076622.						
ε_{11}^a / i		6.0798		2.9364		
ε_{12}^a		6.0725		2.9294		
ε_{22}^a / i		6.0651		2.9224		0.97203
ε_{13}^a / i		0.30067		0.16174		
ε_{32}^a		0.30026		0.16131		-0.82761
ε_{33}^a / i		0.015276		0.0092128		0.85792
A distribution, subject to distortion by LH and EC waves.Normalized current = .0877						
ε_{11}^a / i		5.6372		3.0821		
ε_{12}^a		5.6294		3.0672		
ε_{22}^a / i		5.6217		3.0524		45.75
ε_{13}^a / i		0.29229		0.25282		
ε_{32}^a		0.29165		0.25025		-11.98
ε_{33}^a / i		0.01851		0.03639		4.7856