Ballooning instabilities in tokamaks with low flow shear

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Introduction

Transport barriers in tokamaks create regions of high normalised pressure gradient (α). This can drive instabilities such as the ballooning mode which then limits the pressure and may trigger edge localised modes (ELMs).

In the absence of flow shear (s_v) it has been shown [1] that the most unstable ballooning modes have the toroidal mode number $n \to \infty$. However transport barriers are associated with sheared flows so it is important to investigate the influence of flow on the $n \to \infty$ ballooning modes [2, 3, 4].

We employ a large aspect ratio, circular cross section tokamak $(s-\alpha)$ equilibrium model [5] to explore the stability of a radial perturbation of plasma (F). This is described in ballooning space by the equation [3, 4]:

$$\frac{\partial}{\partial \eta}\left[\left(1+P^2\right)\frac{\partial F}{\partial \eta}\right]+\Gamma F = s_v^2 \frac{\partial}{\partial \tau}\left[\left(1+P^2\right)\frac{\partial F}{\partial \tau}\right] \tag{1}$$

Here η is the ballooning space coordinate, $\tau = s_v t$, t is the time, $s_v = -(d\Omega/dq)$ is the flow shear (with Ω the toroidal flow normalised to the Alfvén frequency), $s = (r/q)(dq/dr)$ is the magnetic shear, $P = s\eta - \alpha \sin(\eta+\tau)$ and $\Gamma = \alpha\left[\cos(\eta+\tau) + P\sin(\eta+\tau)\right]$. For the ballooning transform to exists F must be bounded in η.

Equation (1) has a Floquet solution:

$$F(\eta, \tau) = \hat{F}(\eta, [\tau]) \exp\{\gamma t\} \tag{2}$$

where γ is the Floquet growth rate and square brackets indicate a periodic dependence. A 2-D code has been written which solves the ballooning equation (Eq 1) for the eigenfunction $\hat{F}(\eta, [\tau])$ and growth rate γ (as an eigenvalue).

In the limit $s_v \to 0$, we define $G = F\sqrt{1+P^2}$ where G satisfies:

$$\frac{\partial^2 G}{\partial \eta^2} + VG = s_v^2 \frac{\partial^2 G}{\partial \tau^2} \tag{3}$$

$$V = -\frac{[s-\alpha\cos(\eta+\tau)]^2}{(1+P^2)^2} + \frac{\alpha\cos(\eta+\tau)}{1+P^2} \tag{4}$$

In the limit $|\eta| \to \infty$, where the "potential" $V \sim 1/|\eta|^4$, Eq 3 reduces to a wave equation. Therefore, at a sufficiently large $|\eta|$ general solutions of Eq 3 have the form of incoming and outgoing waves. For G to be bounded in η and growing in time, we must choose the outgoing wave solution $G(\tau - s_v|\eta|)$. This forms our numerical boundary condition in η. Alternatively we may apply the condition $G = 0$ at very large $|\eta|$. We have found that the outgoing wave condition is more convient as it can be applied at lower values of $|\eta|$ than the condition $G = 0$.

At sufficiently low values of s_v the rate at which the amplitude varies (t) is fast compared to the rate at which the potential varies (τ). Therefore we can introduce a WKB (or adiabatic) approximation of the form:

$$G(\eta,t,\tau) = H(\eta,\tau)\exp\left\{\frac{1}{s_v}\int_0^\tau \gamma_0(\tau')\, d\tau'\right\} \tag{5}$$

Then H satisfies the eigenvalue equation:

$$\frac{\partial^2 H}{\partial \eta^2} + VH = \gamma_0^2(\tau) H \tag{6}$$

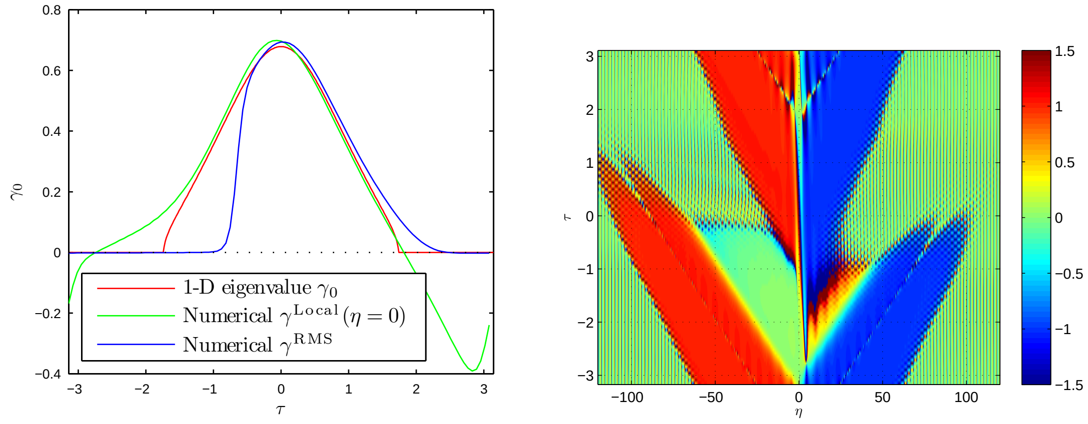
This approach is valid provided $\gamma_0^2(\tau) > 0$, which we refer to as the unstable region. However when $\gamma_0^2 < 0$ there is a continuum of stable solutions and the eigenvalue (γ_0) is not well defined [4]. In fact as $\gamma_0^2 \to 0$ we cannot separate timescales, so this analysis is only valid for times where, $\gamma_0^2(\tau) > \varepsilon$, where $s_v \ll \varepsilon \ll 1$.

Numerical studies

We can define a local growth rate from the numerical solution of the 2-D Eq 3 by evaluating $\gamma^{\text{Local}}(\tau) = (1/G)(\partial G/\partial \tau)$ at some convenient η, for example $\eta = 0$. Another, perhaps more satisfactory, procedure is to define the instantaneous growth rate in terms of the RMS average of G, $< G >$:

$$\langle G\rangle(\tau) = \left[\int_{-\infty}^{\infty} G^2(\eta,\tau)\, d\eta\right]^{\frac{1}{2}} \tag{7}$$

$$\gamma^{\text{RMS}}(\tau) = \frac{1}{\langle G\rangle}\frac{\partial \langle G\rangle}{\partial \tau} \tag{8}$$



(a) (b)

Figure 1: Typical results from the numerical solution to Eq (3) (with $\alpha = 1.4$, $s = 1.0$ and $s_v = 0.05$). (a) Shows plots of the instantaneous (RMS) and local growth rate calculated from the 2-D code and the eigenvalue from the 1-D code. (b) Shows the character of out going waves. For $\eta > 0$, outgoing waves are blue, incoming red and vice versa for $\eta < 0$.

Figure 1a shows a comparison between $\gamma_0(\tau)$, $\gamma^{\mathrm{Local}}(\tau)$ and $\gamma^{\mathrm{RMS}}(\tau)$. The RMS growth rate $\gamma^{\mathrm{RMS}}(\tau)$ is zero throught the stable region and for the early part of the unstable region, but then rises very rapidly, after which it agrees well with $\gamma_0(\tau)$. The local growth rate, $\gamma^{\mathrm{Local}}(\tau)$ is strongly negative at the start of the stable region, but rises to become positive before the end of that region; for most of the unstable region it agrees well with $\gamma_0(\tau)$

To check that we have an outgoing wave solution $G(\tau - s_v|\eta|)$ we can plot $s_v(\partial G/\partial\tau)/(\partial G/\partial\eta)$, which should be -1 when $\eta > 0$ and $+1$ when $\eta < 0$ (figure 1b). For the unstable region of η we find an outgoing wave as expected. However, unexpectedly, in the stable region we find a mixture of both incoming and outgoing waves. There are two possible causes for this that are under investigation: 1) reflections from the η boundary or 2) reflections from the tail of the potential.

Taylor Toy Model

To explore the stable region of τ, we have developed the "Taylor Toy Model" (TTM). In this model the potential V is:

$$V = \delta(\eta)D(\tau) \tag{9}$$

where $D(\tau) = c + \sin\tau$ and $\delta(\eta)$ is a δ function. This equation can be solved exactly giving:

$$G(\eta,\tau) = C\exp\int_0^{\tau - s_v|\eta|}\frac{D(\tau')}{2s_v}d\tau' \tag{10}$$

Analytically $\gamma^{\mathrm{Local}}(\tau)$, which follows $D(\tau)/2$, shows strong damping throughout the stable region and strong growth throughout the unstable region. When calculated numerically,

$\gamma^{\text{Local}}(\tau)$ shows damping in the early part of the stable region, but growth in the later part of this region. In the unstable region it agrees well with $\gamma^{\text{Local}}(\tau)$.

Analytically $\gamma^{\text{RMS}}(\tau)$ shows zero growth in the stable region and in a large part of the unstable region. Numerically it also shows zero growth in the stable region and in a somewhat smaller part of the unstable region.

Note that the time-average of the analytic local and instantaneous (RMS) growth rates are identical (and equal to $c/2$), even though at times there may be large differences between them.

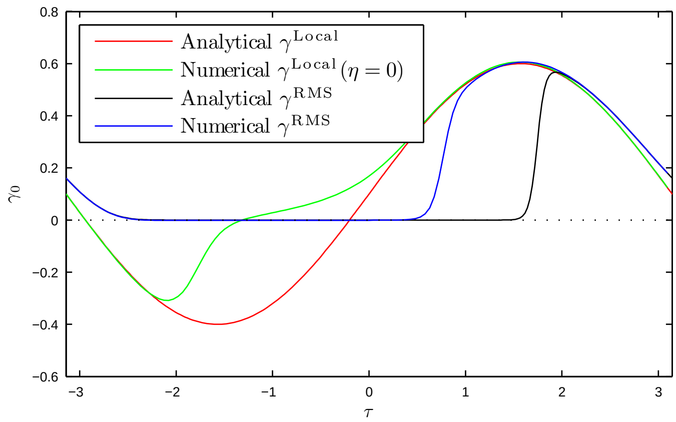


Figure 2: Plots of the instantaneous and local growth rates for the Taylor Toy Model, calculated both analytically and numerically (with $c = 0.2$ and $s_v = 0.05$)

We have found that the Taylor Toy Model is particularly sensitive to reflections at the η boundary and it is these small, spurious reflections that cause the difference between the analytical and numerical growth rates. We are exploring this in more detail but this sensitivity to reflected waves suggests that in the full model the tail in the potential $V\ \left(\sim \eta^{-4}\right)$ may be influential due to wave reflections - despite its very small amplitude.

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