Single null analytic solution to the Grad-Shafranov equation

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I. Introduction

Simple analytic solutions to the Grad-Shafranov (GS) equation are desirable and useful for studying the equilibrium, stability and transport properties of toroidally axisymmetric fusion devices, and for benchmarking magnetohydrodynamics (MHD) equilibrium codes. The simplest analytic solutions are obtained with pressure and current profiles which are linear in the flux function Ψ, the so-called Solov'ev profiles. These solutions have been extensively studied (e.g. [1]), and have given very useful insights, for instance in the study of plasma shaping effects in spherical tokamaks (STs).

Unfortunately, the Solov'ev profiles correspond to the unrealistic situation where the toroidal current has a jump at the plasma edge. The goal of this paper is to show that concise, relatively simple analytic solutions to the GS equation can also be obtained for the physical case where the pressure and current profiles are quadratic in the flux function Ψ so that they, and their surface gradients vanish at the plasma edge. This work extends previous related studies [2] by following the same successful procedure previously demonstrated with Solov'ev profiles [3]. Specifically we present a simple method to construct single null tokamak and ST equilibria allowing for arbitrary aspect ratio, elongation, triangularity and β.

II. Grad-Shafranov equation with quadratic pressure and current profiles

In the usual (R,ϕ,Z) cylindrical coordinate system, where ϕ is the ignorable coordinate in axisymmetric devices, the GS equation [4] is

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)+\frac{\partial^2\Psi}{\partial Z^2}=-\mu_0 R^2\frac{dp}{d\Psi}-\frac{1}{2}\frac{dF^2}{d\Psi} \tag{1}$$

where $2\pi\Psi(R,Z)$ is the poloidal flux, $-2\pi F(\Psi) = I_p(\Psi)$ is the net poloidal current flowing in the plasma and the toroidal field coils, and $p = p(\Psi)$ is the plasma pressure. We introduce the normalization $\Psi = \Psi_0\psi$, $R = R_0 x$, and $Z = R_0 y$, where Ψ_0 is an arbitrary constant, and R_0 is the major radius of the fusion device under consideration, and focus on pressure and current profiles which are quadratic in ψ. Specifically, we write $F^2 = R_0^2 B_0^2\left(1-\alpha\psi^2\right)$, and $p(\psi) = p_0\psi^2$. B_0 is the vacuum magnetic field, α represents the plasma diamagnetism ($\alpha > 0$) or paramagnetism ($\alpha < 0$), and p_0 is the pressure at the magnetic axis. Equation (1) reduces to

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2\psi}{\partial y^2} + k^2\left[1 + b\left(x^2 - 1\right)\right]\psi = 0$$
$$k^2 = \frac{R_0^4 B_0^2}{\Psi_0^2}\left(\frac{2\mu_0 p_0}{B_0^2} - \alpha\right) \qquad b = \left(1 - \frac{\alpha B_0^2}{2\mu_0 p_0}\right)^{-1} \tag{2}$$

Here k is treated as an unknown constant, which is determined from the boundary conditions, while b is chosen according to the regime of operation (for example, $b = 1 \rightarrow \beta = 0$, $b = 0 \rightarrow \beta_p \approx 1$, $b = -1/2\varepsilon \rightarrow \varepsilon\beta \approx 1$). In the regimes of interest, $k^2 > 0$. We solve Eq. (2) by separation of variables, writing $\psi(x,y) = \tilde{\psi}_1(x)\cos\left(k_y y\right) + \tilde{\psi}_2(x)\sin\left(k_y y\right)$, with ${k_y}^2 \le k^2$ an undetermined separation constant. The first term corresponds to up-down symmetric solutions, while the second term corresponds to up-down asymmetric solutions. Also, for $k_y = 0$, $\sin\left(k_y y\right)$ has to be replaced by y, the simplest up-down asymmetric solution. Inserting this expression for $\psi(x,y)$ into Eq. (2) leads to the same ordinary differential equation for $\tilde{\psi}_1(x)$ and $\tilde{\psi}_2(x)$:

$$x\frac{d}{dx}\left(\frac{1}{x}\frac{d\tilde{\psi}}{dx}\right) + \left[k^2\left(1-b\right) - k_y^2 + k^2 b x^2\right]\tilde{\psi} = 0 \tag{3}$$

The general solution to Eq. (3) is: $\tilde{\psi}\left(x\right) = c\,W_{\lambda,\frac{1}{2}}\left(-ik\sqrt{b}x^2\right) + d\,M_{\lambda,\frac{1}{2}}\left(-ik\sqrt{b}x^2\right)$, where W and M are the Whittaker functions, $\lambda = i\left(4k\sqrt{b}\right)^{-1}\left[k^2\left(1-b\right) - k_y^2\right]$. The free constants c and d are determined from the boundary conditions.

Theoretically, specifying the entire continuous shape of the plasma boundary would require an infinite number of free constants c, d, and k_y. However, the following expansion

$$\begin{aligned}\psi(x,y) = W_{\lambda_0} + c_1 M_{\lambda_0} + \left(c_2 W_{\lambda_1} + c_3 M_{\lambda_1}\right)\cos(ky) + \left(c_4 W_{\lambda_2} + c_5 M_{\lambda_2}\right)\cos(k_2 y) \\ + \left(c_6 W_{\lambda_0} + c_7 M_{\lambda_0}\right)y + \left(c_8 W_{\lambda_1} + c_9 M_{\lambda_1}\right)\sin(ky) + \left(c_{10} W_{\lambda_2} + c_{11} M_{\lambda_2}\right)\sin(k_2 y)\end{aligned} \quad (4)$$

leads to a very good match between a desired plasma shape and the actual shape obtained by solving for the finite number of unknown free constants using a corresponding number of boundary constraints. In Eq. (4) the subscript 0 corresponds to $k_y = 0$, the subscript 1 to $k_y = k$, and the subscript 2 to $0 < k_y < k$. The last task is to define the 13 boundary constraints required to determine the 13 free constants $c_1 - c_{11}$, k_y and k.

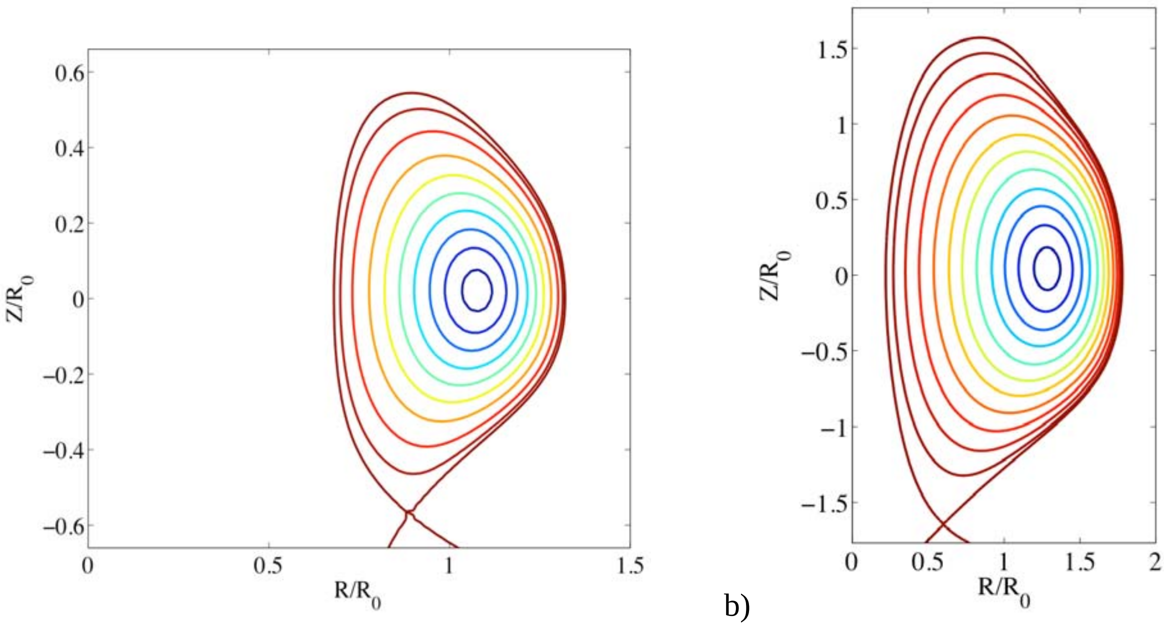
III. Up-down asymmetric solutions and boundary conditions

As in [3], we assume that we have a parametric representation $x(\tau), y(\tau)$ of the desired plasma boundary, where τ is an angle-like variable. Choosing the free additive constant associated with the function ψ so that $\psi = 0$ on the surface, the 13 free constants for a single null divertor tokamak are determined from the following boundary conditions:

$$\begin{aligned}
&\psi(1+\varepsilon,0) = 0 && \psi\left(1-\varepsilon\delta,\kappa\varepsilon\right) = 0 \\
&\psi(1-\varepsilon,0) = 0 && \psi\left[x(\tau = 3\pi/4), y(\tau = 3\pi/4)\right] = 0 \\
&\psi_y(1+\varepsilon,0) = 0 && \psi(x_X, -y_X) = 0 \\
&\psi_y(1-\varepsilon,0) = 0 && \psi_x\left(1-\varepsilon\delta,\kappa\varepsilon\right) = 0 \\
&\psi_{yy}(1+\varepsilon,0) = -N_1\psi_x(1+\varepsilon,0) && \psi_x\left(x_X,-y_X\right) = 0 \\
&\psi_{yy}(1-\varepsilon,0) = -N_2\psi_x(1-\varepsilon,0) && \psi_y\left(x_X,-y_X\right) = 0 \\
&\psi_{xx}(1-\varepsilon\delta,\kappa\varepsilon) = -N_3\psi_y(1-\varepsilon\delta,\kappa\varepsilon) &&
\end{aligned} \quad (5)$$

Here, $\varepsilon = a/R_0$ is the inverse aspect ratio, κ is the upper elongation, δ is the upper triangularity, N_1, N_2, and N_3 are the curvatures of the parametric curve $x(\tau), y(\tau)$ at the outboard midplane, the inboard midplane, and the top respectively. For example, for the well-known model surface used in the examples below, $x(\tau) = 1 + \varepsilon\cos(\tau + \alpha\sin\tau)$, $y(\tau) = \varepsilon\kappa\sin\tau$, with $\sin\alpha = \delta$, we find $N_1 = -(1+\alpha)^2/\varepsilon\kappa^2$, $N_2 = (1-\alpha)^2/\varepsilon\kappa^2$, and $N_3 = -\kappa/\varepsilon\cos^2\alpha$. The parameters x_X and $-y_X$ give the location of the separatrix, and can be chosen freely, although, of course, not every location will lead to an acceptable equilibrium. Often, in particular for small ε, a good choice is $x_X = 1 - 1.05\delta\varepsilon,\ y_X = 1.05\kappa\varepsilon$.

Equation (5) is a system of 13 equations for the 13 unknowns, which is easily solved numerically using typical nonlinear root solvers. In Fig. 1. we show ITER-like and NSTX-like equilibria obtained by this procedure, for a high plasma β corresponding to the vanishing of the toroidal current density gradient at the inboard midplane (i.e. for $b = 1/\varepsilon(2-\varepsilon)$).



a) b)

Fig. 1. a) Lower single null ITER-like equilibrium. $\varepsilon = 0.32, \kappa = 1.7, \delta = 0.33$. b) Lower single null NSTX-like equilibrium. $\varepsilon = 0.78, \kappa = 2, \delta = 0.2, x_X = 0.6, y_X = 1.05\kappa\varepsilon$.

IV. Up-down symmetric solutions and boundary constraints

The procedure also works for up-down symmetric equilibria with both smooth or double null surfaces. The up-down symmetry implies $c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = 0$. The reduced set of 7 boundary constraints for the 7 remaining unknown constants can be found in [3].

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