On steady poloidal and toroidal flows in tokamak plasmas

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1. Introduction

The Grad-Shafranov (G-S) equation can be generalised to take poloidal and toroidal flows into account, with closure provided by an energy equation [1]. In many analyses of this problem an isentropic energy equation has been used, i.e. specific plasma entropy $\sigma = p/\rho^\gamma$ has been assumed to be constant on a given magnetic flux surface. However tokamak plasmas are generally in a weakly collisional regime, with very rapid parallel heat transport and a high level of entropy-creating turbulence, concentrated on the low field side of the plasma. Flux surface variations of electron and ion temperature T_e, T_i are thus likely to be much smaller than those of σ, and it is more appropriate to use an isothermal energy equation. Whereas the G-S equation in the absence of flows is elliptic, "transonic" poloidal flows (of the order of the sound speed c_s times the poloidal magnetic field B_θ divided by the total field B) can render the combined system of G-S and Bernoulli equations hyperbolic in part of the solution domain. Spatial transitions from elliptic to hyperbolic behaviour imply the presence of radial discontinuities, e.g. in density [3]. This possibility is of particular interest in tokamaks, in view of the observed correlation between sheared flows and internal transport barriers (ITBs), which are characterised by steep gradients in temperature and density [2].

2. Flow equilibria with isentropic flux surfaces

Lovelace et al. [1] derived Grad-Shafranov-Bernoulli equations describing axisymmetric MHD flow equilibria, with closure provided by assuming either specific entropy conservation along streamlines, i.e. $\mathbf{v}\cdot\nabla\sigma = 0$, or thermal equilibrium within flux surfaces, i.e. $\mathbf{B}\cdot\nabla(p/\rho) = 0$. Using the Bernoulli relation to express ρ in terms of $\nabla\Psi$ where Ψ is poloidal flux, the terms in the G-S equation involving second order derivatives of Ψ can be written as [1]

$$\left(1-\mu_0\frac{F'^2}{\rho}\right)\left(A_{RR}\frac{\partial^2\Psi}{\partial R^2}+A_{RZ}\frac{\partial^2\Psi}{\partial R\partial Z}+A_{ZZ}\frac{\partial^2\Psi}{\partial Z^2}\right), \quad (1)$$

where $F = F(\Psi)$ is defined such that the poloidal momentum density is $\nabla F \times \nabla\varphi$, (R,φ,Z) being right-handed cylindrical coordinates and, in the isentropic case,

$$A_{RR} = 1-\frac{v_\theta^2 v_Z^2}{c_s^2 c_{A\theta}^2-(c_s^2+c_A^2)v_\theta^2+v_\theta^4}, \quad A_{RZ} = \frac{2v_\theta^2 v_R v_Z}{c_s^2 c_{A\theta}^2-(c_s^2+c_A^2)v_\theta^2+v_\theta^4}, \quad (2)$$

$$A_{ZZ} = 1 - \frac{v_\theta^2 v_R^2}{c_s^2 c_{A\theta}^2 - (c_s^2 + c_A^2)v_\theta^2 + v_\theta^4}. \qquad (3)$$

Here v_R and v_Z are the radial and vertical components of the poloidal flow v_θ, $c_A = B/(\mu_0\rho)^{1/2}$ and $c_{A\theta} = B_\theta/(\mu_0\rho)^{1/2}$. The G-S equation is elliptic (hyperbolic) if $D \equiv A_{RZ}^2 - 4A_{RR}A_{ZZ}$ is negative (positive). It is straightforward to show that

$$D = -4\frac{1 - M_{s\theta}^2(1+\beta)}{1 - M_{s\theta}^2(1+\beta) + \beta M_{s\theta}^4(B_\theta/B)^2}, \qquad (4)$$

where $M_{s\theta} = (v_\theta/c_s)(B/B_\theta)$ and $\beta = \gamma\mu_0 p/B^2$. It follows from Eq. (4) that the G-S equation is hyperbolic for $M_{s\theta}$ in the range [4]

$$\frac{1}{1+\beta} < M_{s\theta}^2 < \frac{(1+\beta)}{2\beta}\frac{B^2}{B_\theta^2}\left[1 - \left(1 - \frac{4\beta}{(1+\beta)^2}\frac{B_\theta^2}{B^2}\right)^{1/2}\right]. \qquad (5)$$

For $\beta \ll 1$, $B_\theta \ll B$, as in Fig. 1(a), the G-S equation is hyperbolic for only a very narrow range of values of $M_{s\theta}$ [5]. However the ordering $\beta(B_\theta/B)^2 \ll 1$ is not necessarily satisfied on the low field side of spherical tokamak (ST) plasmas, where B_θ/B and β can both be of order unity. In this case the G-S equation is hyperbolic at lower $M_{s\theta}$ and for a greater range of values of this parameter [Fig. 1(b)]. The presence of transonic poloidal flows in ST plasmas with $\beta \sim 1$, $B_\theta \sim B$ could pose serious numerical challenges for equilibrium reconstruction.

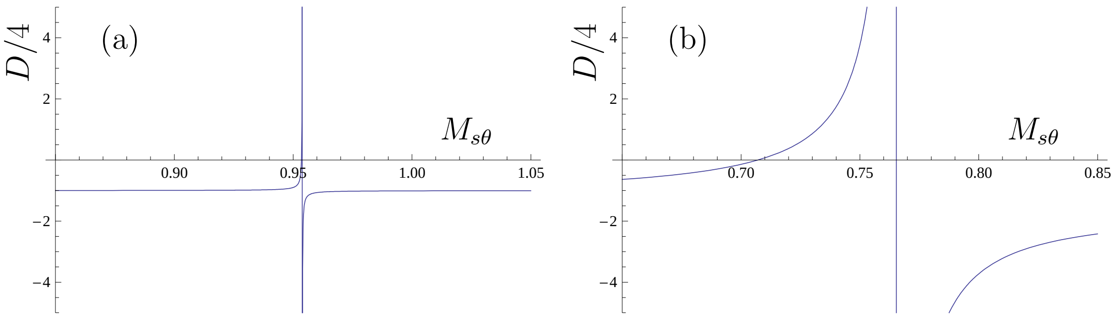

Fig. 1 Discriminant of 2nd order derivatives in the G-S equation versus $M_{s\theta}$ for (a) $\beta = 0.1$, $B_\theta = 0.1B$, and (b) $\beta = 1$, $B_\theta = B$. The equation is hyperbolic when $D > 0$.

3. Flow equilibria with isothermal flux surfaces

Results obtained using the isentropic energy equation carry over to the isothermal case, with $v_i = (2T/m_i)^{1/2}$ replacing $c_s = (\gamma p/\rho)^{1/2}$. Eqs. (4-5) remain valid, with $\beta = \mu_0 p/B^2$ and $M_{s\theta} = (v_\theta/v_i)(B/B_\theta)$ [6]. When $\beta \ll 1$, Eq. (5) shows that the threshold v_θ for the G-S equation to become hyperbolic is given by $M_{s\theta} = 1$; this is a factor $\gamma^{1/2} \simeq 1.3$ lower in the isothermal case than in the isentropic case (a significant reduction, if the actual poloidal flows are close to the threshold). When toroidal and poloidal flows v_φ, v_θ are present there exists a flux function [6]

$$\Omega(\Psi) = \frac{v_\varphi}{R} - \frac{v_\theta}{R}\frac{B_\varphi}{B_\theta}. \qquad (6)$$

When flux surfaces are isothermal the appropriate form of the Bernoulli relation is

$$H(\Psi) = \frac{2T}{m_i} \ln\left(\frac{\rho}{\rho_0}\right) + \frac{v_\varphi^2 + v_\theta^2}{2} - \Omega R v_\varphi. \tag{7}$$

Eliminating v_φ from Eqs. (6) and (7) we obtain an expression for number density $n \simeq \rho/m_i$:

$$n = n_1(\Psi) \exp\left[\frac{\Omega^2(\Psi)}{2v_i^2(\Psi)}\left(R^2 - R_0^2\right) - \frac{1}{2}M_{s\theta}^2\right]. \tag{8}$$

Ω, v_i and $M_{s\theta}$ can be measured on the low field side of the plasma using neutral beam diagnostics (charge exchange and motional Stark effect). Because of beam attenuation and injection geometry, such measurements are generally not possible on the high field side. Thomson scattering data can be used to infer n and T_e across the midplane; measurements of inboard/outboard density asymmetry, combined with Eq. (8), can then yield $M_{s\theta}$ on the high field side. Moreover Eq. (6) can be used to determine v_φ on the high field side, provided that $B_\theta^2 \ll B^2$. The above expressions could thus provide a more complete picture of global tokamak dynamics.

Expressing B_φ in terms of Ψ, R and n we find that Eq. (8) can be written as a transcendental equation for normalised density $x = n/n_0$ (n_0 being an arbitrary constant density):

$$x = x_0 \exp\left[-\frac{M_s^2}{2x^2}\left\{1 + \frac{\xi^2 x^2}{(x - M_{A\theta}^2)^2}\right\}\right] \equiv x_0 g_1(x), \;\; x_0 = \frac{n_1(\Psi)}{n_0} \exp\left[\frac{\Omega^2}{2v_i^2}\left(R^2 - R_0^2\right)\right]. \tag{9}$$

Here $M_s = F' B_\theta/(m_i n_0 v_i)$, $M_{A\theta} = F'(\mu_0/m_i n_0)$ and $\xi \sim B_\varphi/B_\theta$. Fig. 2(a) shows $g_1(x)$ and $g_2 \equiv x/x_0$ in the limit $M_s \ll 1$, $M_{A\theta} \ll 1$, $\xi \gg 1$. There is a critical x_0 above which two solutions exist for the density, i.e. two values of x such that $g_2 = g_1$. These correspond to subsonic ($M_{s\theta} < 1$) and supersonic ($M_{s\theta} > 1$) poloidal flows: similar solutions have been found in the isentropic case [5]. For any set of values of M_s, $M_{A\theta}$ and ξ there are values of x_0 such that four solutions exist: two additional solutions, represented by the first two intersections of the dashed-dotted line with the solid curve in Fig. 2(b), correspond to $v_\theta \simeq c_{A\theta}(1 + \xi^{2/3})$, i.e. trans-Alfvénic flows.

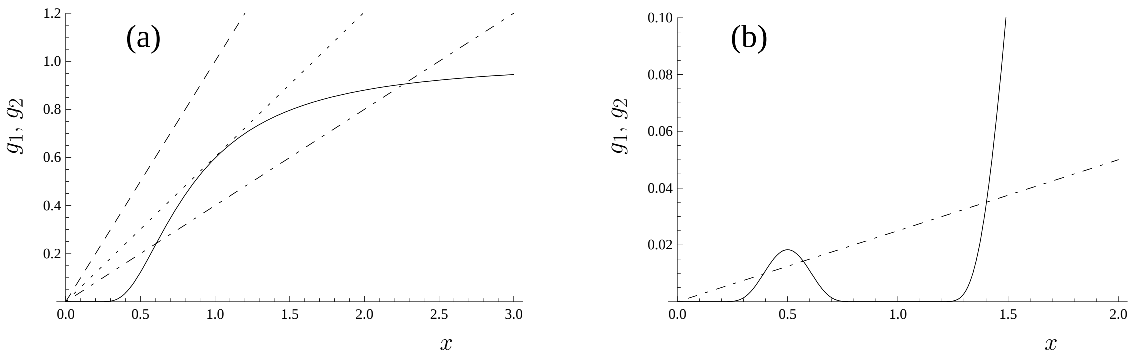

Fig. 2 (a) $g_1(x)$ for $M_s = M_{A\theta} = 0.1$, $\xi = 10$ (solid curve) and $g_2(x)$ for $x_0 = 1.0$ (dashed), 1.65 (dotted), 2.5 (dashed-dotted). (b) $g_1(x)$ for $M_s = M_{A\theta} = 1$, $\xi = 1$ (solid curve) and $g_2(x)$ for $x_0 = 40$ (dashed-dotted). Density solutions correspond to points at which $g_2 = g_1$.

4. Comparison with experiment

The highest v_θ yet measured in JET was $75 \pm 20\,\text{kms}^{-1}$ (in the ion diamagnetic direction), inside a transport barrier in shot 58094 [2]. Immediately outside the barrier, v_θ was close to zero. These measurements, combined with inferred values of T_i, n and q, suggest that $M_{s\theta}$ was of order unity inside the barrier, falling with increasing minor radius to a value close to zero outside the barrier. Abrupt falls in n and p were thus accompanied by a fall in Mach number. In contrast, solutions of the G-S equation with isentropic flux surfaces show discontinuous drops in n associated with *rises* in $M_{s\theta}$, from clearly subsonic to clearly supersonic values [5]. Eq. (8) suggests that this type of relation would also be expected in the isothermal case. We conclude that there is no evidence in these data of radial discontinuities, although the fact that $M_{s\theta}$ was of order unity suggests that density profiles were significantly affected by poloidal flows [cf. Eq. (8)]. This conclusion applies *a fortiori* to poloidal flow measurements in the MAST ST, which are broadly consistent with low values expected from neoclassical theory [7].

5. Conclusions

The threshold poloidal flow for the G-S equation to be hyperbolic is lower when flux surfaces are isothermal rather than isentropic, and the range of flows for which the equation is hyperbolic is greater in high performance STs than in conventional tokamaks. An expression for the density variation on a flux surface in the presence of poloidal and toroidal flows could be used to infer experimental information about both types of flow on the high field side of tokamak plasmas, where direct measurements are generally not possible. The Bernoulli relation for isothermal flux surfaces has four solutions for the density, corresponding to different flow regimes. There is no clear evidence of any regime other than the subsonic one being realised in present-day tokamaks, but it may be possible to access other regimes in future experiments.

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