Verification of noniterative equilibria reconstruction against analytical solutions of the Grad-Shafranov equation

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A noniterative reconstruction approach for magnetic equilibria has been proposed [1, 2] which, instead of assuming some predefined model for the two free functions $P(\psi)$ and $F(\psi)$ in the Grad-Shafranov (GS) equation,

$$-R^2\nabla\cdot\left(R^{-2}\nabla\psi\right) = \mu_0 R^2 \dot{P}(\psi) + F(\psi)\dot{F}(\psi), \tag{1}$$

builds its own models in order to ensure that the reconstructed pressure and vertical-field profiles match those derived from the experimental data taken along a given chord in terms of laboratory coordinates. For convenience, these can be the set (r,θ,ϕ) which is related with the cylindrical coordinates (R,Z) as $R = R_0 - r\cos\theta$ and $Z = r\sin\theta$, where R_0 is the major axis, θ is a poloidal angle measured clockwise from the midplane high-field side, and ϕ is the toroidal angle. Without going into further details, extensively described elsewhere [1, 2], the solution to the GS equation is assembled as a mixed power and trigonometric series

$$\psi(r,\theta;\varepsilon) = \psi_{00}(r) + \sum_{n=1}^{N}\frac{\varepsilon^n}{n!}\sum_{k=0}^{n}\psi_{nk}(r)\cos k\theta, \tag{2}$$

truncated at the N-th power of the inverse aspect ratio ε, thus containing at most N harmonics only, and where all radial functions $\psi_{nk}(r)$ are solutions of the same linear ODE,

$$r^2\psi''_{nk}(r) + r\psi'_{nk}(r) + \left[s(r) - k^2\right]\psi_{nk}(r) = b_{nk}(r), \tag{3}$$

but with different source terms $b_{nk}(r)$. In practice, choosing the truncation order allows one to balance the computational cost with the intended accuracy, while ensuring an unique solution for a given set of boundary conditions. Such possibility becomes quite handy when fast (albeit somewhat less accurate) equilibria are needed but conventional solvers stall or converge too slowly. Here, the noniterative algorithm is tested against an analytical solution of the GS equation in order to assess its convergence with increasing truncation order, as well as the computational costs involved. The verification is made with plasma parameters typical of the ITER baseline design [3], asserting that the asymptotic *ansatz* (2) is not restricted to the limits $\varepsilon \to 0$ and $\beta \to 0$ (with β the usual plasma beta), which are seldom of practical use, but is able to handle also magnetic equilibria attained in fusion-grade tokamak devices.

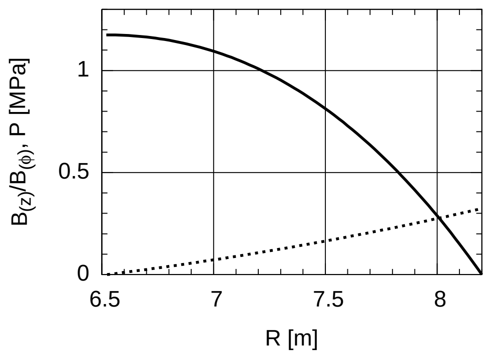

Figure 1: Input magnetic-field pitch (dotted) and pressure (solid) profiles.

Despite their simplicity, Solovev equilibria [4] have long since played a pivotal role whenever analytical solutions of the GS equation are required: theoretical studies of tokamak equilibria, stability, and transport properties, and benchmarking of numerical codes. Setting the flux derivatives $\dot{P}(\psi)$ and $F(\psi)\dot{F}(\psi)$ to some constant values turns (1) into a linear non-homogeneous PDE and its solution splits into the sum of a particular solution with a linear combination of solutions of the homogeneous equation $\nabla\cdot\left(R^{-2}\nabla\psi\right)=0$ which can be suitably tailored in order to achieve a plasma surface with a prescribed shape [5]. In this work, the solution

$$\psi_{\mathrm{S}}(R,Z)=\frac{(A-1)}{8}R^4+\frac{A}{2}R^2\ln R+\sum_{i=1}^{7}C_i\psi_i(R,Z) \tag{4}$$

will be used, where $\psi_i(R,Z)$ are some special solutions of the homogeneous problem and the constants C_i are computed so the plasma surface follows, as close as possible, the shape $R(\tau)=R_0+a\cos\left(\tau+\alpha\sin\tau\right)$ and $Z(\tau)=a\kappa\sin\tau$, with a, κ, and δ being, respectively, the minor radius, the elongation, and the triangularity, while $\delta=\sin\alpha$ and A is set by the β value [5].

To test the noniterative approach, one first needs to find the several constants that enable (4) to describe an equilibrium with ITER-like parameters [3]: $a=2$m, $R_0=6.2$m, $\kappa=1.7$, $\delta=0.33$, $I_{\mathrm{p}}=15$MA, and $B_0=5.3$T. The last two values, i.e., the plasma current and the on-axis toroidal field, are needed to find A from the requirement that $\beta_{\mathrm{T}}=0.05$, yielding a normalized value $\beta_{\mathrm{N}}=3.5\%$, which is already close to the MHD stability limit [3]. Next, the analytical solution is used to infer the experimental data that diagnostics would report if the plasma equilibrium was to be described by $\psi_{\mathrm{S}}(R,Z)$ and which one needs to feed into the noniterative algorithm. This data set reduces to the pressure and magnetic-field pitch distributions throughout a single line starting at the magnetic axis and running along the high-field side of the plasma midplane (Figure 1), together with values of the poloidal flux evaluated at a few points (more precisely, $N-1$) of the poloidal section. Such poloidal-flux values are used to extract the boundary condi-

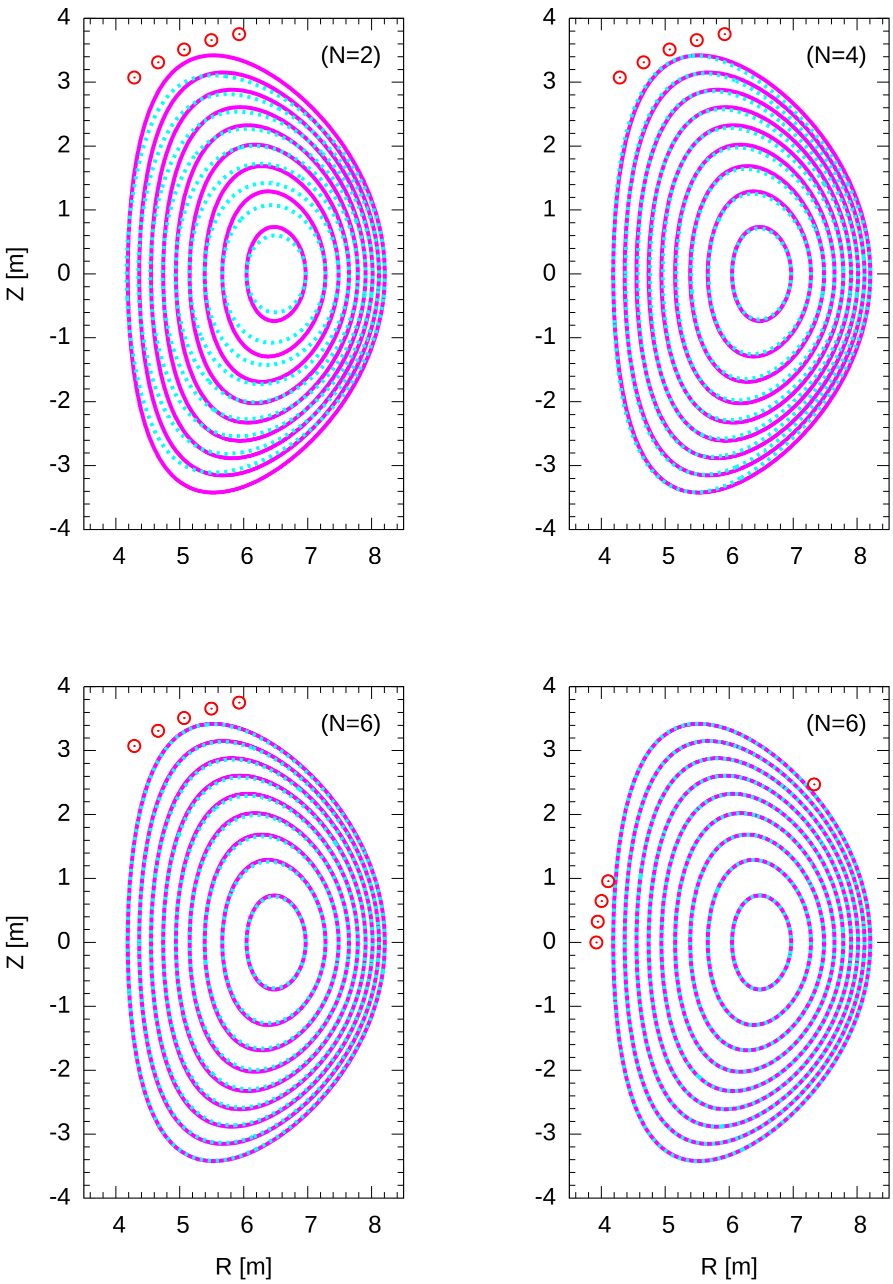

Figure 2: Flux surfaces of the analytical solution (solid lines) and of the numerical equilibria (dotted lines) for $N = 2, 4,$ and 6. The poloidal flux is picked-up at the red dots.

tions required when each function $\psi_{nk}(r)$ is computed from (3) and, to keep the approach indeed noniterative and single-pass, they all must be picked up along a circumference of arbitrary radius but centred at the origin $r = 0$. The results of the reconstruction procedure are depicted in Figure 2, showing the convergence towards $\psi_S(R,Z)$ with increasing truncation order N. It shows also that the constraint over poloidal-flux pick up allows still for suitable locations to be chosen, outside the plasma volume but close enough to the boundary. Finally, the average error

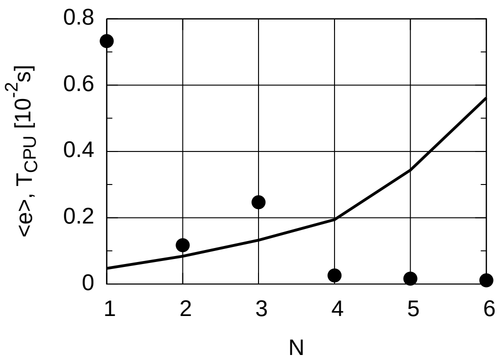

Figure 3: Average error (dots) and CPU time required per reconstruction (solid line).

$\langle e\rangle = (\int|\psi_S|\,dRdZ)^{-1}\int|\psi-\psi_S|\,dRdZ$ and the CPU time required to compute each numerical equilibrium in a single 2.53GHz core are plotted in Figure 3 versus the truncation order N.

In conclusion, the noniterative approach for equilibria reconstruction has been successfully tested and verified against a Solovev-type solution. Although the latter is somewhat oversimplified, it must be stressed that its only purpose was to provide the input data to the algorithm. The numerical solutions have been shown to converge towards the analytic target for ITER-like plasma parameters, this requiring about 6ms on a standard CPU. Moreover, the ODEs yielding each $\psi_{nk}(r)$ for a given n and $k = 1,\ldots,n$ are fully decoupled, making it easy to profit from the parallel computing power of multicore nodes and get even faster solutions.

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