The Bennett Pinch revisited

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Abstract

The original derivation of the well-known Bennett relation is presented. Willard H. Bennett developed a theory, considering both electric and magnetic fields within a pinched column, which is completely different from that found in the textbooks. The latter theory is based on simple magnetohydrodynamics which ignores the electric field.

1. An elementary theory of the pinch effect

The configuration of the pinch effect is illustrated in Fig. 1. A current flowing in the axial direction interacts with a self-produced magnetic field to produce an inwardly directed $\underline{j} \times \underline{B}$ force. If we adopt a magnetohydrodynamic model for the equilibrium case we can write

$$\nabla p = \underline{j} \times \underline{B} \tag{1}$$

and

$$\oint \underline{B} \cdot \underline{dl} = \mu_0 \int \underline{j}.\underline{dS} \tag{2}$$

The first equation describes the force balance, and the second equation is Ampères Law, the integral form of the Maxwell equation

$$\nabla \times \underline{B} = \mu_0 \underline{j}.$$

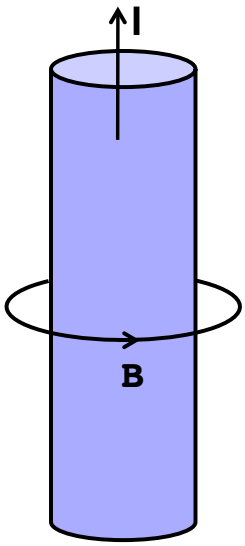

Figure 1: The pinch effect

In the present case equations 1 and 2 take the form

$$\frac{dp}{dr} = -jB \tag{1a}$$

and

$$2\pi rB = \mu_0 I(r) \tag{2a}$$

where $I(r)$ is the current flowing within a radius r. It can readily be shown that these two equations lead to the following result , whatever the current distribution $I(r)$.

$$2Nk(T_e + T_i) = (\mu_0/4\pi) I^2 \tag{3}$$

This result is known as the Bennett relation. N is the number of electrons or ions per unit length of the plasma column, I is the total current, k is Boltzmann's constant and μ_0 is the permeability of free space, $\mu_0 = 4\pi x10^{-7}$ H/m.

2. The theory developed by Bennett (1934)

The model studied by Bennett [1], illustrated in Fig. 2, was entirely different from the MHD model described in the previous paragraph. Bennett assumed that the electrons have a Maxwellian distribution with a drift u in the z direction, the drift being independent of the radius r. The positive ions have a velocity v in the $-z$ direction.

In a co-ordinate system moving with the electrons they do not experience a $\underline{j} \times \underline{B}$ force, since there is no electron current in this system. Their radial distribution will be given by the Boltzmann relation,

$$n_{11} = n_{110} \exp\left[\frac{eV_1}{kT_{11}}\right]. \tag{4}$$

The notation used here is that subscript (1) refers to electrons and (2) refers to ions. A second subscript refers to the co-ordinate system in which the density is measured. In a co-ordinate system moving with the ions the ion density is given by

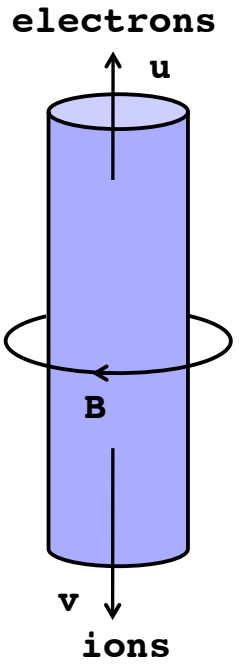

Figure 2: The Bennett pinch.

$$n_{22} = n_{220} \exp\left[-\frac{eV_2}{kT_{22}}\right]. \tag{5}$$

In the first system of co-ordinates, that moving with the electron drift velocity, Poisson's equation takes the form

$$\nabla^2 V_1 = -\frac{e}{\varepsilon_0}\left[n_{21} - n_{11}\right],$$

where n_{21} is the ion density in that system. Using eqn. 4 we can write

$$\nabla^2 \log n_{11} = \frac{e^2}{\varepsilon_0 k T_{11}}\left[n_{11} - n_{21}\right]. \tag{6}$$

Analogously

$$\nabla^2 V_2 = -\frac{e}{\varepsilon_0}\left[n_{22} - n_{12}\right],$$

so that

$$\nabla^2 \log n_{22} = \frac{e^2}{\varepsilon_0 k T_{22}}\left[n_{22} - n_{12}\right]. \tag{7}$$

Let $\alpha_1 = e^2/\varepsilon_0 kT_{11}$ and $\alpha_2 = e^2/\varepsilon_0 kT_{22}$. The equations can be transformed into equations relating to quantities measured in the rest system of co-ordinates, using well-known results from the special theory of relativity.

$$n_{11} = \frac{n_1}{\gamma_1};\; n_{12} = \gamma_2 \left[1 + \frac{uv}{c^2}\right] n_1;\; n_{21} = \gamma_1 \left[1 + \frac{uv}{c^2}\right] n_2;\; n_{22} = \frac{n_2}{\gamma_2}$$

where $\gamma_1 = \left(1 - u^2/c^2\right)^{-1/2}$ and $\gamma_2 = \left(1 - v^2/c^2\right)^{-1/2}$. Equations 6 and 7 then become

$$\nabla^2 \log n_1 = \alpha_1 \gamma_1 \left[1 - \frac{u^2}{c^2}\right] n_1 - \alpha_1 \gamma_1 \left[1 + \frac{uv}{c^2}\right] n_2 \tag{8}$$

and

$$\nabla^2 \log n_2 = \alpha_2 \gamma_2 \left[1 - \frac{v^2}{c^2}\right] n_2 - \alpha_2 \gamma_2 \left[1 + \frac{uv}{c^2}\right] n_1. \tag{9}$$

We can note that the temperatures are also relativistically transformed, $T_1 = \gamma_1 T_{11}$ and $T_2 = \gamma_2 T_{22}$. Strictly speaking the temperatures employed by Bennett are "two-dimensional" temperatures, the relevant velocities are those transverse to the drift velocities of the electrons and ions. The following simple solution to the pair of equations 8 and 9 was found by Bennett

$$n_1 = \frac{n_0}{\left[1 + bn_0 r^2\right]^2} \tag{10a}$$

and

$$n_2 = \frac{\alpha_1 \gamma_1 \left[1 - \frac{u^2}{c^2}\right] + \alpha_2 \gamma_2 \left[1 + \frac{uv}{c^2}\right]}{\alpha_2 \gamma_2 \left[1 - \frac{v^2}{c^2}\right] + \alpha_1 \gamma_1 \left[1 + \frac{uv}{c^2}\right]} n_1, \tag{10b}$$

where

$$b = \frac{\alpha_1 \alpha_2 \gamma_1 \gamma_2}{8c^2} \frac{(u+v)^2}{\alpha_1 \gamma_1 \left[1 + \frac{uv}{c^2}\right] + \alpha_2 \gamma_2 \left[1 - \frac{v^2}{c^2}\right]}. \tag{10c}$$

We can note that the quantity b has the dimensions of length. When the electron drift velocity becomes comparable with the velocity of light we no longer have a plasma in that the electron and ion densities are no longer closely equal. One consequence of this is that we can no longer employ a single fluid model as in magnetohydrodynamics.

The solution found is relativistically invariant. It is of interest, however, to find a very good approximate solution for the case where the velocities u and v are small compared with the velocity of light c. The solution then becomes

$$n_1 = n_2 = \frac{n_0}{\left[1 + bn_0 r^2\right]^2}, \tag{11}$$

where

$$b = \frac{\mu_0 e^2 (u+v)^2}{8k(T_1 + T_2)},$$

We can note that n_1 and n_2 are very nearly equal, but there must be a small difference to produce the electric field given by Poisson's equation. The Lorentz factors γ_1 and γ_2 introduced in order to obtain equations 8 and 9 are *not* identically equal to unity. The total number of electrons, or ions, per unit length is readily found by integration to be $N = \pi/b$ so that

$$N = \frac{8\pi k\,(T_1 + T_2)}{\mu_0 e^2\,(u+v)^2}$$

and

$$\frac{\mu_0 I^2}{4\pi} = 2Nk\,(T_1 + T_2)\,, \tag{12}$$

which is the well-known Bennett relation.

In practice the current will be mainly carried by electrons, i.e. v is very small. We can note that the combination of perpendicular electric and magnetic fields does *not* lead to an $\underline{E} \times \underline{B}$ drift of the ions of magnitude E/B as stated in some textbooks.

If v is negligible, then in the rest system of coordinates we can write

$$n_2 = n_{20} \exp\left[-\frac{eV}{kT_2}\right]. \tag{13}$$

The conclusion is that the positive ions are confined by the electrostatic field, and they are not confined by orbiting around magnetic field lines. This conclusion will no doubt apply to all "magnetic confinement" systems. This leads to the interesting question as to whether the possibility of purely electrostatic confinement should be seriously reconsidered. It is known that such a system can exist in thermal equilibrium, a natural state of matter, whereas magnetically confined plasmas are inherently unstable.

References

[1] W.H. Bennett, *Magnetically Self-focusing Streams*, Phys. Rev. **45**, 890 (1934)