Particle interactions with spatially localized wavepackets

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Wave-particle interaction is one of the most well studied subjects in plasma physics. Particle dynamics in the presence of electrostatic or electromagnetic waves has been one of the main paradigms on which the modern theory of nonlinear Hamiltonian dynamics and chaos has been applied [1]. However, almost all previous studies of wave-particle interactions from the point of view of Hamiltonian dynamics have been focused on waves having discrete spectra, namely, periodic waves. In a previous work [2] we have studied particle interactions with localized wavepackets propagating in the absence of a magnetic field or along a uniform magnetic field. In this work we study such interactions in the general case where the wavepackets propagate at an angle to a magnetic field.

The Hamiltonian describing particle motion in a uniform constant magnetic field $\mathbf{B}=B_0\mathbf{z}$ is $H_0=|\mathbf{p}-(e/c)\mathbf{A}|^2/(2M)$, where $\mathbf{A}=-B_0 y\mathbf{x}$ is the vector potential corresponding to the magnetic field and $\mathbf{p}=M\mathbf{v}+(e/c)\mathbf{A}$ is the canonical momentum. Utilizing the generating function $F_1=M\Omega[(y-Y)^2\cot\varphi/2-xY]$ ($\Omega=eB_0/Mc$, is the gyration frequency) we transform to "guiding center" variables with the new Hamiltonian being $H_0 = P_z^2/2M + P_\varphi\Omega$. The new variables are the guiding center position (X,Y), the z coordinate and momentum (z,P_z) and the gyration angle and angular momentum (φ, P_φ). Under the presence of a localized electrostatic wave field $\Phi_0(\mathbf{r}-\mathbf{V_g}t)\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)$, with wavenumber $\mathbf{k} = k_\parallel\mathbf{z}+k_\perp\mathbf{y}$ and group velocity $\mathbf{V_g}$, the Hamiltonian is

$$H = \frac{P_z^2}{2M} + P_\varphi\Omega + e\Phi_0\left(X-V_xt+\rho\cos\varphi, Y-V_yt-\rho\sin\varphi, z-V_zt\right)\sum_m J_m \sin(k_\parallel z + k_\perp Y - m\varphi - \omega t)$$

where $\rho(P_\varphi) = (2P_\varphi/M\Omega)^{1/2}$ is the Larmor radious and $J_m(k_\perp\rho)$ are Bessel functions. The wave fields range from ordinary wavepackets to ultra short few-cycle and subcycle transient pulses. Note that for the latter, the assumption of adiabaticity for amplitude modulation, commonly adopted in previous works, does not hold. We consider the presence of the localized wave as a perturbation to the particle motion in the constant uniform magnetic field. The unperturbed frequencies of particle motion are $\omega_\varphi = \Omega$ and $\omega_z = P_z/M (= v_z)$. The perturbation results in resonances between the degrees of freedom given by the condition

$$k_\parallel \frac{P_z}{M} - \omega = m\Omega$$

In order to analyze particle dynamics we utilize the canonical perturbation method [1]. According to this method we construct a near-identity canonical transformation resulting to a new Hamiltonian where the dependence on canonical positions is "pushed" to higher order. According to a standard procedure [1], the first order generating function is calculated by integrating the perturbative part of the Hamiltonian along unperturbed particle orbits. Although our approach is general, in the following we focus on a localized wave of Gaussian form

$$\Phi_0(x,y,z) = \exp\left(-\frac{x^2}{a_\perp} - \frac{y^2}{a_\perp} - \frac{z^2}{a_\parallel} \right)$$

In order to consider finite Larmor effects, i.e. take into account the fact that the Larmor radious can be comparable to the spatial with of the wave, we use a first order, with respect to $(\rho/a_\perp)$, Taylor expansion of Φ_0. Therefore we obtain the first order generating function as

$$w_1 = \frac{eE}{2A} \exp\left(\frac{B^2 - 4A^2C^2}{4A^2} \right) \sum_m \exp\left(-\frac{\Omega_m^2}{4A^2} \right) \exp\left(i\Psi_m + i\frac{B\Omega_m}{2A^2} \right)$$

$$\left\{ \left[J_m + \frac{\rho}{a_\perp} R_\perp \left(J_{m-1} e^{-i\theta_1} - J_{m+1} e^{i\theta_1} \right) \right] K\left(At + B_m \right) \right.$$

$$\left. + \frac{\rho}{a_\perp} \frac{V_\perp}{A} \left(J_{m-1} e^{-i\theta_2} - J_{m+1} e^{i\theta_2} \right) \left[B_m K\left(At + B_m \right) - \Lambda\left(At + B_m \right) \right] \right\}$$

where:

$$\mathbf{R} = \left(\frac{X}{a_\perp}, \frac{Y}{a_\perp}, \frac{z - (P_z/M)t}{a_\parallel} \right), \quad \mathbf{V} = \left(\frac{V_x}{a_\perp}, \frac{V_y}{a_\perp}, \frac{V_z - (P_z/M)}{a_\parallel} \right),$$

$$A^2 = |\mathbf{V}|^2, \quad B = -2\mathbf{R}\cdot\mathbf{V}, \quad C^2 = |\mathbf{R}|^2,$$

$$\theta_1 = \tan^{-1}\frac{X}{Y}, \quad \theta_2 = \tan^{-1}\frac{V_x}{V_y}, \quad R_\perp = \sqrt{X^2 + Y^2}\big/a_\perp, \quad V_\perp = \sqrt{V_x^2 + V_y^2}\big/a_\perp,$$

$$\Psi_m = k_\parallel \left[z - (P_z/M)t \right] + k_\perp Y - m\left(\varphi - \Omega t \right), \quad \Omega_m = k_\parallel(P_z/M) - m\Omega - \omega, \quad B_m = \frac{B + i\Omega_m}{2A}$$

The nonperiodic time dependence of the generating function is given through the functions

$$K(t) = \int_{-\infty}^{t} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\left[1 + \text{erf}(t) \right], \quad \lim_{t\to+\infty} K(t) = \sqrt{\pi}$$

$$\Lambda(t) = \int_{-\infty}^{t} x e^{-x^2} dx = -\frac{1}{2} e^{-t^2}, \quad \lim_{t\to+\infty} \Lambda(t) = 0$$

The magnitude of w_1 depends exponentially on the momentum parallel to the magnetic field P_z, the gyration frequency Ω and the wave frequency ω, through the exponential term

$\exp\left(-\Omega_m^2/4A^2\right)$. The generating function has a significant magnitude in the phase space area localized around the locations where the resonance conditions $\Omega_m = 0$ are fulfilled, with the width of these areas being inversely proportional to the transit time of the localized field through the particle ($A \sim V/a$). The effect of the "cross-section" of the wave-particle scattering is taken into account through the exponential term

$$\exp\left(\frac{B^2 - 4A^2C^2}{4A^2}\right) = \exp\left(-\frac{|\mathbf{R} \times \mathbf{V}|^2}{|\mathbf{V}|^2}\right)$$

First-order finite Larmor radius effects are taken into account through terms proportional to $(\rho/\alpha_\perp)$.

Having calculated w_1, we can construct first-order approximate invariants $\left(\overline{P}_z, \overline{P}_\varphi, \overline{X}\right)$ of the particle motion as follows:

$$\overline{P} = P - \frac{\partial w_1}{\partial Q}$$

where $(P, Q) = (P_z, z)$, (P_φ, φ), $(M\Omega X, Y)$ are the respective pairs of canonically conjugate variables. By setting two variable pairs equal to constants the contour plots of each one of these approximate invariants provides analytically the Poincare surface of section in the plane defined by the third pair. Moreover, we can study the maximum canonical momentum variation after an interaction of the particle with the localized wave through the equation

$$\overline{P} = P - \max_{z,\varphi,Y}\left(\frac{\partial w_1^\infty}{\partial Q}\right)$$

where $w_1^\infty = \lim\limits_{t \to +\infty} w_1$.

The above results refer to the study of single particle dynamics under interaction with the localized wave field. Based on the generalized Madey's theorem [3], [4], we can utilize the results of first-order perturbation theory in order to calculate position averaged quantities, depending on the canonical momenta, with up to second-order accuracy. Therefore the averaged canonical momentum variation of an ensemble of particles having different initial gyration angles, z and Y positions, after a single interaction with the localized wave, is

$$\left\langle(\Delta P)_2\right\rangle_{z_0,\varphi_0,Y_0} = \frac{1}{2}\frac{\partial}{\partial P_0}\left(\left\langle\left(\frac{\partial w_1^\infty}{\partial Q_0}\right)^2\right\rangle_{z_0,\varphi_0,Y_0}\right)$$

where we have substitute the initial values of the canonical variables in the r.h.s. This equation provides the canonical momentum variation depending on the initial values of P_z, P_φ

and *X.* Additional averaging over the initial guiding center coordinate *X* results in space-averaged parallel and angular momentum variations that are of interest to calculations on energy transfer through wave-particle interactions with applications to heating and current drive in magnetized plasmas.

In addition we can study the transient diffusion of particle momentum and *X* position for an ensemble of particles interacting with the localized wave field by utilizing the following evolution equation for the particle distribution function averaged over (z_0, φ_0, Y_0)

$$\frac{\partial F(\mathbf{P})}{\partial t} = \frac{\partial}{\partial \mathbf{P}}\left(\mathbf{D}(\mathbf{P},t)\frac{\partial F(\mathbf{P})}{\partial \mathbf{P}} \right)$$

where $\mathbf{P} = (P_z, P_\varphi, X)$ and $\mathbf{Q} = (z, \varphi, Y)$ and

$$\mathbf{D}(\mathbf{P},t) = \frac{1}{2}\frac{\partial}{\partial t}\left\langle \left(\frac{\partial w_1}{\partial \mathbf{Q}_0} \right)\left(\frac{\partial w_1}{\partial \mathbf{Q}_0} \right) \right\rangle_{\mathbf{Q}_0}$$

is the time-dependent diffusion tensor [5].

In conclusion, we have studied particle interaction with spatially localized electrostatic waves in a constant uniform magnetic field. The localized fields may range from ordinary wavepackets to ultra short few-cycle and subcycle transient pulses, since no adiabaticity assumption is considered. The utilization of the canonical perturbation theory allowed for the construction of approximate invariants of the motion containing all the essential information for the strongly inhomogeneous phase space of the system, corresponding to the chaotic scattering and transient momentum variation of the particles. Moreover, the collective particle behaviour has been studied by calculating position-averaged momentum variations and transient momentum and position diffusion.

References

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