Plasma physics in non-inertial frames

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1. Introduction

Historically, the equations of plasma physics have generally been studied using inertial reference frames, in which the equations of Newton and Maxwell have their simplest form. However, tokamak plasmas often rotate at velocities comparable to the ion thermal speed [1], while much higher rotation rates are encountered in some astrophysical plasmas [2] and inertially-confined fusion plasmas are subject to extreme accelerations [3]. The observed behaviour of such systems is likely to be better understood by considering the equations determining their evolution in suitable non-inertial frames. We have formulated the Lorentz, Vlasov, guiding centre and fluid equations in an arbitrary accelerating frame, and applied this analysis to a toroidally-rotating tokamak plasma [4,5]. Brizard [6] derived the gyrokinetic equation for such a plasma using an effective magnetic field that depends on particle velocity. Our analysis provides the basis for an alternative approach to gyrokinetic theory in non-inertial frames, whereby particle motions are referred to an effective magnetic field that is independent of particle velocity.

2. Charged particle motion in non-inertial frames

Neglecting the emission and absorption of radiation, the action for a particle of rest mass m and charge Ze in an electromagnetic field A_μ can be written in the covariant form

$$S = -\int_{\tau_{\rm in}}^{\tau_{\rm fin}} [m v_\mu v^\mu + ZeA_\mu v^\mu] d\tau, \tag{1}$$

where $\tau_{\rm in}$, $\tau_{\rm fin}$ denote proper times and v^μ is the particle four-velocity. In Minkowski spacetime $v^\mu = \gamma(c, \mathbf{v})$ where $\gamma = (1-v^2/c^2)^{-1/2}$ and $A_\mu = (\Phi/c, -\mathbf{A})$ where Φ and $\mathbf{A}$ are the electrostatic and magnetic vector potentials; the action then reduces to

$$S = \int_{t_{\rm in}}^{t_{\rm fin}} \mathscr{L} dt, \tag{2}$$

where $t_{\rm in}$, $t_{\rm fin}$ denote coordinate times and $\mathscr{L} = -mc^2(1-v^2/c^2)^{1/2} + Ze(\mathbf{v}\cdot\mathbf{A} - \Phi)$ is the Lagrangian. Requiring that the particle trajectory between $t_{\rm in}$ and $t_{\rm fin}$ be such that S has a stationary value (Hamilton's principle of least action), we obtain the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial v_i}\right) = \frac{\partial \mathscr{L}}{\partial x_i}, \tag{3}$$

where x_i is a general space coordinate and $v_i = dx_i/dt$. We now let $\mathbf{v}$ and $\mathbf{V}$ denote nonrelativistic particle velocities in the laboratory frame and a frame moving at nonrelativistic but otherwise arbitrary velocity $\mathbf{u}_f$ relative to the laboratory, so that $\mathbf{u}_f = \mathbf{v} - \mathbf{V}$. The manifest invariance of $A_\mu v^\mu$ indicates that $\mathbf{A} \cdot \mathbf{v} - \Phi$ is invariant in the nonrelativistic limit. Since A_μ is a four-vector, the potentials in the flow and laboratory frames are related by the expressions

$$\mathbf{A}_f = \mathbf{A}_{\text{lab}}, \qquad \Phi_f = \Phi_{\text{lab}} - \mathbf{A}_{\text{lab}} \cdot \mathbf{u}_f, \tag{4}$$

and the frame-invariant Lagrangian can be written as

$$\mathscr{L} = \frac{1}{2} m V^2 + Ze\mathbf{V} \cdot \left[\mathbf{A}_f + \frac{m}{Ze}\mathbf{u}_f\right] - Ze\left[\Phi_f - \frac{mu_f^2}{2Ze}\right] \equiv \frac{1}{2} m V^2 + Ze\mathbf{V} \cdot \mathbf{A}_* - Ze\Phi_*, \tag{5}$$

where

$$\mathbf{A}_* = \mathbf{A}_f + \frac{m}{Ze}\mathbf{u}_f, \qquad \Phi_* = \Phi_f - \frac{mu_f^2}{2Ze}. \tag{6}$$

Substituting Eq. (5) into Eq. (3) we obtain the equations of motion

$$m\frac{d\mathbf{V}}{dt} = Ze\left(\mathbf{E}_* + \mathbf{V} \times \mathbf{B}_*\right), \tag{7}$$

where the effective fields are

$$\mathbf{E}_* = -\nabla\Phi_* - \frac{\partial \mathbf{A}_*}{\partial t} = \mathbf{E}_f - \frac{m}{Ze}\frac{\partial \mathbf{u}_f}{\partial t} + \frac{m}{2Ze}\nabla u_f^2, \tag{8}$$

$$\mathbf{B}_* = \nabla \times \mathbf{A}_* = \mathbf{B}_f + \frac{m}{Ze}\mathbf{W}_f. \tag{9}$$

Here $\mathbf{W}_f = \nabla \times \mathbf{u}_f$ is the vorticity of the frame flow, and $\mathbf{E}_f$, $\mathbf{B}_f$ are the electric and magnetic fields in the flow frame. In the nonrelativistic limit $\mathbf{B}_f = \mathbf{B}_{\text{lab}} \equiv \mathbf{B}$. There is a generalized Coriolis force, which does no work on the particle, a generalized centrifugal force and a pseudo-force associated with the time-dependence of the frame flow.

The above analysis can be applied to a frame whose velocity is spatially uniform but changing in time. In the absence of electromagnetic fields Eq. (7) then reduces to

$$\frac{d\mathbf{V}}{dt} = -\frac{\partial \mathbf{u}_f}{\partial t}. \tag{10}$$

The frame acceleration provides an effective gravity that drives a Rayleigh-Taylor instability in inertially-confined plasmas [3]. For a frame rotating at a uniform and constant rate ω we have $\mathbf{u}_f = \omega \times \mathbf{R}$, where $\mathbf{R}$ is position in the rotating frame, $\mathbf{W}_f = 2\omega$ and

$$m\frac{d\mathbf{V}}{dt} = Ze\left[\mathbf{E}_f + \frac{m}{Ze}\nabla\left(\frac{\omega^2 R^2}{2}\right) + \mathbf{V} \times \left(\mathbf{B} + \frac{2m}{Ze}\omega\right)\right]. \tag{11}$$

If $\mathbf{B}$ is uniform the velocity-dependent force vanishes in a frame rotating at $\omega = -ZeB/2m \equiv -\Omega_c/2$ (because $\mathbf{B}_* = 0$) and in a frame rotating at $\omega = -\Omega_c$ (because $\mathbf{V} = \mathbf{0}$).

3. Kinetic and guiding-centre equations in non-inertial frames

In the absence of collisions, a particle distribution function in the moving frame f satisfies the Vlasov equation [4]

$$\frac{\partial f}{\partial t} + \mathbf{V}\cdot\nabla f + \frac{Ze}{m}\left(\mathbf{E}_* + \mathbf{V}\times\mathbf{B}_*\right)\cdot\frac{\partial f}{\partial \mathbf{V}} = 0. \tag{12}$$

This can be used to obtain reduced kinetic (e.g. drift-kinetic, gyrokinetic) and fluid descriptions of the plasma, respectively by averaging over particle orbits and by taking velocity-space moments. In the absence of collisions the guiding centre velocity in the moving frame $\mathbf{V}_{gc}$ satisfies

$$m\frac{d\mathbf{V}_{gc}}{dt} = Ze\mathbf{V}_{gc}\times\mathbf{B}_* + Ze\mathbf{E}_* - \mu_*\nabla B_*, \tag{13}$$

where μ_* is magnetic moment defined in terms of gyromotion around $\mathbf{B}_*$.

4. Application to rotating tokamak plasmas

Using Eq. (13) it can be shown that heavy impurity ions in transonically-rotating tokamak plasmas are centrifugally trapped on the low field side of the plasma, with a bounce frequency [5]

$$\omega_b \simeq \omega\left(\frac{r}{Rq^2}\right)^{1/2}\left(1 - \frac{Zm_iT_e}{m_Z(T_e+T_i)}\right)^{1/2} \equiv \frac{\varepsilon^{1/2}\omega_*}{q}, \tag{14}$$

where m_i is bulk ion mass, T_e and T_i are electron and bulk ion temperatures, $\varepsilon = r/R$ is inverse aspect ratio, q is safety factor and m_Z, Ze denote impurity ion mass and charge. Moreover the effective flux surfaces for low charge states of heavy impurity ions can differ significantly from laboratory frame flux surfaces [5], implying better confinement of impurity ions in plasmas rotating counter to the plasma current than in those rotating in the co-current direction (Fig. 1).

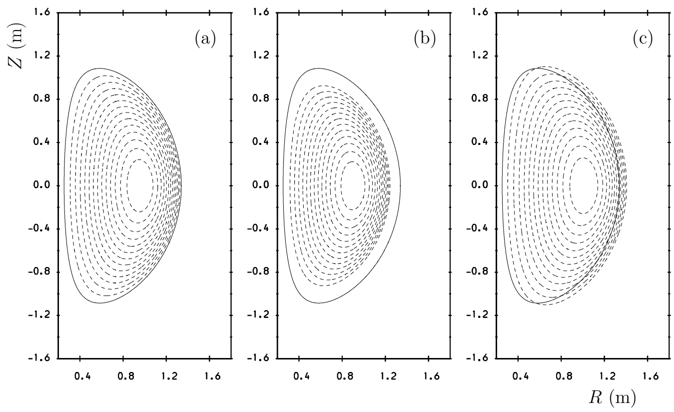

Fig. 1 Effective flux surfaces for W^{20+} ions in (a) stationary, (b) counter-rotating and (c) co-rotating MAST-like plasmas.

Considering momentum balance for impurity ions in a frame co-rotating with bulk ions, one can infer a radially-outward advection velocity [5]

$$v_R \simeq \frac{e^2 m_i^{1/2} m_Z q^2 \omega_*^2 R n \ln\Lambda}{6\sqrt{2}\pi^{3/2}\varepsilon_0^2\varepsilon^2 B^2 T_i^{3/2}}\left[1 + 2\frac{q}{\varepsilon}\frac{\omega}{\Omega_Z}\right]^{-2}, \tag{15}$$

in the usual notation, where Ω_Z is impurity ion cyclotron frequency. This result is consistent with test-particle simulations of collisional impurity transport (Fig. 2), and may have important implications for the retention of fusion-relevant impurities such as tungsten.

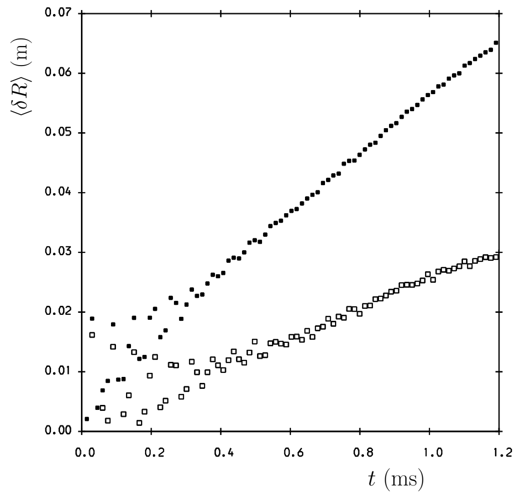

Fig. 2 Mean excursion of impurity ions with $A = 184$, $Z = 20$ (solid squares) and $A = 92$, $Z = 10$ (open squares) in simulation of test-particle orbits in counter-rotating MAST-like plasma.

5. Conclusions

The equation of motion of a charged particle can be written in identical form in non-relativistic inertial and non-inertial frames, with the effective electromagnetic fields in the latter depending on both the Maxwell fields and the frame flow $\mathbf{u}_f$. Generalized Coriolis and centrifugal forces introduce additional terms $(m/Ze)\mathbf{u}_f$ and $-mu_f^2/(2Ze)$ in the effective vector and scalar potentials. In a toroidally-rotating tokamak plasma the Coriolis force shifts the effective flux surfaces while the centrifugal force enhances particle trapping on the low field side. Vlasov, gyrokinetic, guiding centre and fluid equations can be derived as in an inertial frame, using effective fields and potentials. The analysis can also be extended to include relativistic effects, making it applicable to astrophysical plasmas such as pulsar magnetospheres [2].

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